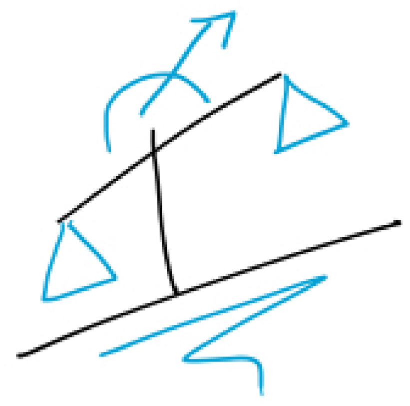


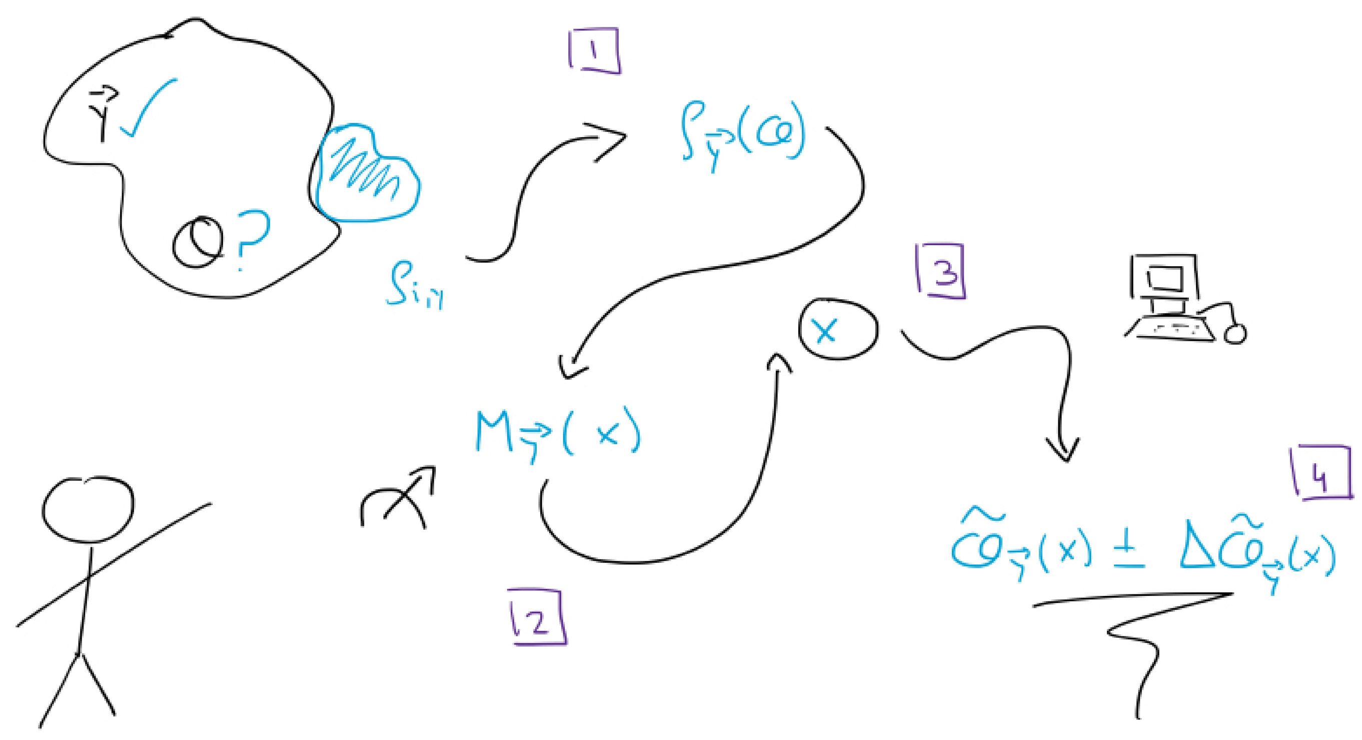
From quantum thermometry to the quantum estimation of scales, and back

Jesús Rubio*

Department of Physics and Astronomy, University of Exeter, Stocker Road, Exeter EX4 4QL, UK

Overview

- The framework of global quantum thermometry is decoupled from its thermodynamic origin and extended as to accommodate the estimation of any scale in physics—temperature or otherwise.
- Upon reintroducing the language of equilibrium thermometry, scale estimation is shown to establish the optimality of energy measurements regardless of the initial temperature range.
- This result bridges the gap between current approaches to Bayesian thermometry and the full power of quantum estimation theory.
- More generally, quantum scale estimation completes a trio of metrological frameworks for three elementary quantities in physics: phases, locations, and scales.



Quantum scale estimation

Specification

Let an experiment be described by:

- a dimensionless measurand x,
- a set of *known* parameters $\boldsymbol{y} = (y_1, y_2, \dots)$, and
- a parameter Θ which is *unknown*.

What is a scale parameter?

We say that Θ scales y_i if, for fixed Θ, y_i is considered 'large' when $y_i/\Theta \gg 1$ and 'small' when $y_i/\Theta \ll 1$. The key aspect of this definition is its invariance under transformations

$$y_i \mapsto y_i' = \gamma y_i, \ \ \Theta \mapsto \Theta' = \gamma \Theta,$$

with positive γ, since $y_i/\Theta = y_i'/\Theta'$.

Metrological protocol

1. Prepare the hypothesis-encoded state $\rho_{\boldsymbol{y}}(\theta)$
2. Implement the POM $M_{\boldsymbol{y}}(x)$
3. Record the outcome x, whose statistics is given by the Born rule
$$\mathrm{Tr}[M_{\boldsymbol{y}}(x)\rho_{\boldsymbol{y}}(\theta)] = h\left(x, \frac{\boldsymbol{y}}{\theta}\right)$$
4. Calculate a scale estimator $\tilde{\theta}_{\boldsymbol{y}}(x) \pm \Delta\tilde{\theta}_{\boldsymbol{y}}(x)$

Optimisation problem

$$\min_{M_{\boldsymbol{y}}(x), \tilde{\theta}_{\boldsymbol{y}}(x)} \mathrm{Tr}\left\{ \int dx\, M_{\boldsymbol{y}}(x)\, W_{\boldsymbol{y}}[\tilde{\theta}_{\boldsymbol{y}}(x)] \right\},$$

where

$$W_{\boldsymbol{y}}[\tilde{\theta}_{\boldsymbol{y}}(x)] = \int d\theta\, p(\theta)\, \rho_{\boldsymbol{y}}(\theta) \log^2 \left[\frac{\tilde{\theta}_{\boldsymbol{y}}(x)}{\theta} \right]$$

and $p(\theta)$ is a prior probability.

Minimum mean logarithmic error

$$\bar{\epsilon}_{\mathrm{mle}} \geq \bar{\epsilon}_p - \mathcal{K}_{\boldsymbol{y}} \geq \bar{\epsilon}_p - \mathcal{J}_{\boldsymbol{y}},$$

where

- $\bar{\epsilon}_p$ only depends on the prior probability,
- the first inequality is saturated upon using the optimal scale estimator, and
- the second inequality is saturated upon using, in addition, the optimal POM.

The expressions for $\bar{\epsilon}_p$, $\mathcal{K}_{\boldsymbol{y}}$, and $\mathcal{J}_{\boldsymbol{y}}$ are given in Refs. [1,2].

Optimal quantum strategy

Let the operator

$$\mathcal{S}_{\boldsymbol{y}} = \int ds\, \mathcal{P}_{\boldsymbol{y}}(s)\, s$$

solve the Lyaponuv equation

$$\mathcal{S}_{\boldsymbol{y}} \varrho_{\boldsymbol{y},0} + \varrho_{\boldsymbol{y},0} \mathcal{S}_{\boldsymbol{y}} = 2\varrho_{\boldsymbol{y},1},$$

where

$$\varrho_{\boldsymbol{y},k} = \int d\theta\, p(\theta) \rho_{\boldsymbol{y}}(\theta) \log^k \left(\frac{\theta}{\theta_u} \right);$$

then,

- the optimal estimator is
$$\tilde{\theta}_{\boldsymbol{y}}(x) \mapsto \tilde{\vartheta}_{\boldsymbol{y}}(s) = \theta_u \exp(s),$$
- and the optimal POM is
$$M_{\boldsymbol{y}}(x) \mapsto \mathcal{M}_{\boldsymbol{y}}(s) = \mathcal{P}_{\boldsymbol{y}}(s).$$

This completely solves the problem of estimating scale parameters using quantum states.

Revisiting global quantum thermometry

Equilibrium thermometry

Consider the thermal state

$$\rho_{\boldsymbol{y}}(\theta) = \frac{\exp[-H/(k_B\theta)]}{\mathrm{Tr}\{\exp[-H/(k_B\theta)]\}} = \sum_n |n\rangle\langle n|\, h_n\left(\frac{\boldsymbol{y}}{\theta}\right),$$

where $H = \sum_n |n\rangle\langle n|\, \varepsilon_n$ is a Hamiltonian, $\boldsymbol{y} = (\varepsilon_0, \varepsilon_1, \dots)/k_B$ are the known parameters, and

$$h_n\left(\frac{\boldsymbol{y}}{\theta}\right) = \frac{\exp(-y_n/\theta)}{\sum_m \exp(-y_m/\theta)}$$

are scale-invariant components.

Optimal measurement strategy for an arbitrary prior $p(\theta)$

Using the new toolbox, one finds

$$\mathcal{S}_{\boldsymbol{y}} = \sum_n |n\rangle\langle n|\, \frac{\chi_{\boldsymbol{y}}^{n,1}}{\chi_{\boldsymbol{y}}^{n,0}},$$

where

$$\chi_{\boldsymbol{y}}^{n,k} = \int d\theta\, p(\theta)\, \frac{\exp(-y_n/\theta) \log^k(\theta/\theta_u)}{\sum_m \exp(-y_m/\theta)}.$$

The optimal strategy is thus to measure energy.

Remarks

This calculation...

- ...*proves that energy measurements can be optimal for arbitrarily large prior temperature ranges*
- ...demonstrates that quantum scale estimation can generate practical optimal strategies

For a perspective on alternative formulations of Bayesian thermometry [3–6], see Refs. [1,2].

For the application of this approach to *release-recapture thermometry*, see Ref. [7] and its poster.

Discussion: extending the scope of quantum metrology

Dedicated quantum estimation theories depending on the type of parameter

Type of parameter	phase	location	scale
General support	$0 \leq \theta < 2\pi$	$-\infty < \theta < \infty$	$0 < \theta < \infty$
Symmetry	$\theta \mapsto \theta' = \theta + 2\gamma\pi,\ \gamma \in \mathbb{Z}$	$\theta \mapsto \theta' = \theta + \gamma,\ \gamma \in \mathbb{R}$	$\theta \mapsto \theta' = \gamma\theta,\ \gamma \in \mathbb{R}_{++}$
Maximum ignorance	$p(\theta) = 1/2\pi$	$p(\theta) \propto 1$	$p(\theta) \propto 1/\theta$
Deviation function $\mathcal{D}(\tilde{\theta}, \theta)$	$4\sin^2[(\tilde{\theta} - \theta)/2]$	$(\tilde{\theta} - \theta)^2$	$\log^2(\tilde{\theta}/\theta)$

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* J.Rubio-Jimenez@exeter.ac.uk

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