Quantum Fokker-Planck master equation for continuous feedback control

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Measurement and feedback is a valuable tool for controlling microscopic systems [1]. In this work, we derive a quantum Fokker-Planck master equation decsribing the joint system-detector dynamics under continuous measurement and feedback. For fast measurements, we find a Markovian master equation for the system alone. This equation can describe nonlinear feedback, going beyond the master equation by Wiseman and Milburn [2]. It further implies fluctuation theorems, highlighting the connection between thermodynamics and information theory.

1. Main result

- General setup

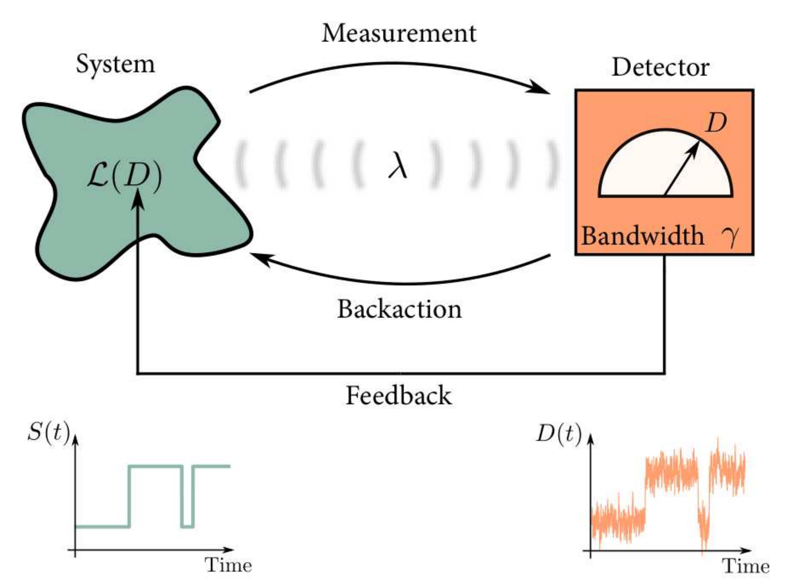


Figure 1. Illustration of a generic measurement and feedback setup, consisting of an open quantum system and a detector with finite bandwidth γ. The detector is continuously measuring an arbitrary system observable $\hat{A}$. The measurement strength λ parameterizes measurement backaction and measurement uncertainty. Continuous feedback is applied using the measurement outcome D to control the Liouville superoperator $\mathcal{L}(D)$ of the system. The time traces visualize trajectories for the system state $S(t)$ and the measurement record $D(t)$.

- Quantum Fokker-Planck master equation

The following master equation describes the joint system-detector time evolution under continuous measurement and feedback control,

$$\partial_t\hat\rho_t(D) = \mathcal{L}(D)\hat\rho_t(D) + \lambda\mathcal{D}[\hat A]\hat\rho_t(D) - \gamma\partial_D\mathcal{A}(D)\hat\rho_t(D) + \frac{\gamma^2}{8\lambda}\partial_D^2\hat\rho_t(D) \qquad (1)$$

- ▶ $\hat\rho_t(D)$ joint system-detector density operator
- ▶ $\hat\rho_t = \int dD\hat\rho_t(D)$ system state for unknown outcome D
- ▶ $P_t(D) = \mathrm{tr}\{\hat\rho_t(D)\}$ probability distribution detector
- ▶ First term: $\mathcal{L}(D)$ feedback controlled time evolution of system. Can describe nonlinear feedback protocols.
- ▶ Second term: dephasing due to measurement backaction, where

$$\mathcal{D}[\hat A]\hat\rho \equiv \hat A\hat\rho\hat A - \frac{1}{2}\{\hat A^2, \hat\rho\}$$

- ▶ Last two terms: Fokker-Planck equation for detector time evolution, following an Ornstein-Uhlenbeck process with superoperator drift coefficient

$$\mathcal{A}(D)\hat\rho \equiv \frac{1}{2}\{\hat A - D, \hat\rho\}$$

and diffusion constant $\gamma/8\lambda$.

2. Separation of timescales

Assuming that γ is the largest parameter of Eq. (1), the detector quickly reaches and remains in steady state with respect to the Fokker-Planck terms. Under this condition, the system evolves, to first order in $1/\gamma$, as

$$\partial_t\hat\rho_t = \left[\mathcal{L}_0 + \lambda\mathcal{D}[\hat A] + \gamma^{-1}\mathcal{L}_{\rm corr}\right]\hat\rho_t \qquad (2)$$

The zeroth order correction is obtained by approximating the system-detector density operator, in the eigenbasis $\{|a\rangle\}$ of $\hat A$, as

$$\hat\rho_t(D) = \left[\sum_{aa'} G_{aa'}(D)\mathcal{V}_{aa'}\right]\hat\rho_t$$

with steady state Ornstein-Uhlenbeck distribution and projection superoperator

$$G_{aa'}(D) = \sqrt{4\lambda/\pi\gamma}\,e^{-(4\lambda/\gamma)[D-(\xi_a+\xi_{a'})/2]^2} \qquad \mathcal{V}_{aa'}\hat\rho \equiv \langle a|\hat\rho|a'\rangle\,|a\rangle\langle a'|$$

Here $\hat A|a\rangle = \xi_a|a\rangle$. The zeroth order Liouville superoperator reads

$$\mathcal{L}_0 = \int dD\mathcal{L}(D)\left[\sum_{aa'} G_{aa'}(D)\mathcal{V}_{aa'}\right]$$

- ▶ Equation (2) can describe nonlinear feedback protocols, going beyond the Markovian master equation derived by Wiseman and Milburn [2].
- ▶ Zeroth order contribution implies fluctuation theorems, highlighting the connection between thermodynamics and information theory, see Sec. 4.

3. Outline derivation

- Generalized quantum measurement

$$\hat\rho_t(z) = \hat K(z)\hat\rho_t\hat K^\dagger(z) \qquad \hat K(z) = \left(\frac{2\lambda\delta t}{\pi}\right)^{1/4} e^{-\lambda\delta t(z-\hat A)^2} \quad [3]$$

The measurement operator $\hat K(z)$ describes how the state changes when outcome z is observed. The timestep between measurements is δt. A weak continuous measurement is obtained through repeated measurements, taking $\lambda\delta t\to 0$.

- Bandwidth

The bandwidth γ is introduced as a low-pass frequency filter,

$$D(t) = \int_{-\infty}^{t} ds\gamma e^{-\gamma(t-s)}z(s)$$

such that the detector outcome $D(t)$ is a smoothened version of $z(t)$.

- Feedback

Using the detector outcome $D(t)$, we control the time evolutions between measurements via $\mathcal{L}(D)$.

4. Toy models

In this section, we apply Eqs. (1) and (2) on two toy models, highlighting the use of our formalism.

- Classical model
- ▶ Two-level system coupled to bath which can excite and de-excite the system, see inset Fig. 2. Environmental noise suppresses coherence such that only classical, stochastic dynamics matter.
- ▶ When an excitation is observed, the levels are flipped, extracting the excitation energy Δ.
- ▶ Steady state power, to zeroth order in separation of timescales,

$$P = \Gamma\Delta\Big[(1-\eta)n_B(\Delta) - \eta[n_B(\Delta)+1]\Big] \qquad (3)$$

$$\eta = [1-\mathrm{erf}(2\sqrt{\lambda/\gamma})]/2 \qquad n_B(\Delta) = [\exp(\Delta/k_BT)-1]^{-1}$$

(error probability) (average occupation bath)

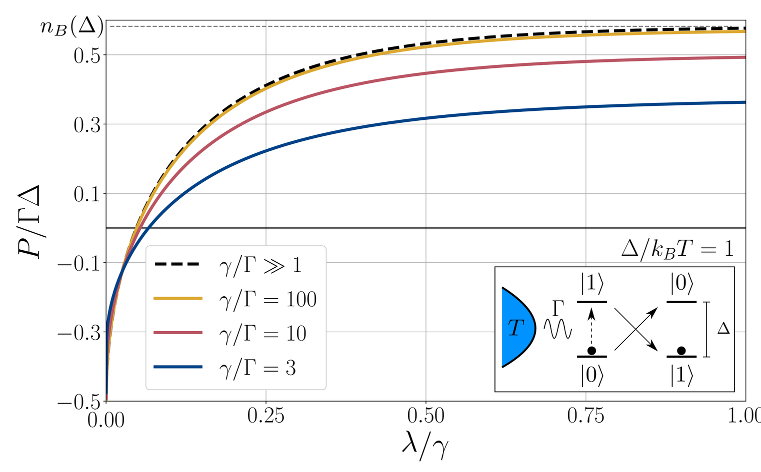


Figure 2. Steady state power production of the classical toy model. Dashed line correspond to Eq. (3). Solid, colored lines were calculated numerically. For strong measurements, the error probability goes to zero, and feedback is always applied correctly. In this regime, the average occupation of the bath limits the power. For weak measurements, error probability goes to 1/2, and feedback is applied randomly, leading to dissipation of energy.

- ▶ Fluctuation theorem relating the probabilities of extracting and dissipating m energy quanta from bath,

$$\frac{P(-m)}{P(m)} = e^{m\left[\Delta/k_BT - \ln\left(\frac{1-\eta}{\eta}\right)\right]}$$

Information term $\ln[(1-\eta/\eta)]$, entropy production of measurement and feedback mechanism. Only information during change of system state matters.

- Quantum model
- ▶ Qubit coherently driven by external driving field, see inset of Fig. 3.
- ▶ Identical measurement and feedback as in classical model. Feedback Hamiltonian given by

$$\hat H_t(D) = [1-\theta(D)]\Delta\,|1\rangle\langle 1| + \theta(D)\Delta\,|0\rangle\langle 0| + g\cos(\Delta t)\hat\sigma_x$$

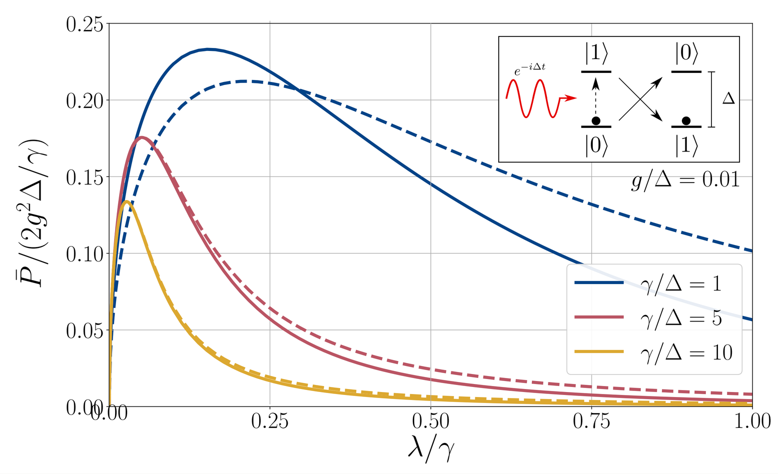
- ▶ Steady state power to first order in separation of timescales, averaged over one driving period,

$$\bar P = \frac{2g^2\Delta}{\gamma}D_0\frac{\Delta^2}{\Delta^2+4\tilde\lambda^2} \qquad (4)$$

$$D_0 = 2\sqrt{\lambda/\pi\gamma}\;{}_2F_2(1/2,1/2;3/2,3/2;-4\lambda/\gamma) \qquad \tilde\lambda = \lambda + \Delta^2\ln(2)/2\gamma$$

(generalized hypergeometric function) (effective dephasing rate)

Figure 3. Steady state power of the quantum model. Dashed lines corresponds to Eq. (4). Solid lines were calculated numerically. For strong measurements, the Zeno effect prevents coherent evolution and the power vanishes. For weak measurements, power vanishes due to random feedback and symmetric driving. Maximum power determined by trade-off between information gain and the Zeno effect.



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