Photon Cooling: Linear vs Nonlinear

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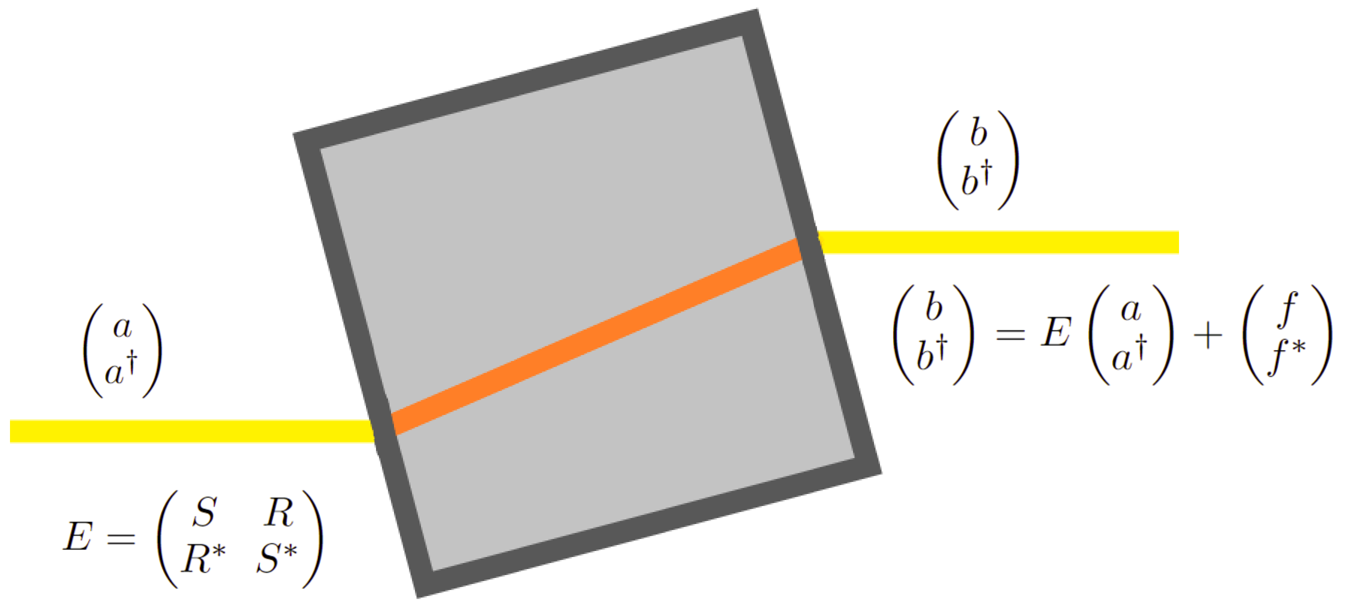
N Mode Linear Evolution



Boson Commutators:

$$S^\dagger S - R^T R^* = I, \quad S^\dagger R = R^T S^*$$

Initial Conditions for Heating:

$$\langle a_j \rangle \equiv \mathrm{tr}(\rho a_j) = 0 \qquad \langle a_i a_j \rangle = 0$$

For an arbitrary mode the **amount of photons** is defined as $\langle a_i^\dagger a_i \rangle$. For the initial conditions provided above linear evolution results in

$$\sum_{i=1}^{N} \left(\langle b_i^\dagger b_i \rangle - \langle a_i^\dagger a_i \rangle \right) \geq 0$$

$$\langle \Delta a^2 \rangle \equiv \frac{1}{2} \langle a a^\dagger + a^\dagger a \rangle - |\langle a \rangle|^2 = \langle a^\dagger a \rangle + \frac{1}{2} - |\langle a \rangle|^2$$

We can see that not only the total amount of photons in the system as a whole but also the noise (variation of the creation / annihilation operators for a mode) get larger which is why we interpret this effect as **heating**. This is an irreversible process that is akin to the **second law of thermodynamics** but independent from it; it holds for linear evolution only but for more general initial conditions.

Bimodal System: COP and Efficiency

Following the definition of heating we can define cooling as the inverse effect; the decrease in the amount of photons in the total system that is in an equilibrium state. It is important to note that this cooling considers **the total amount of the photons in the system** which is different from the usual definition of energy cooling; cooling a subsystem but heating the equilibrium system as a whole.

$$\rho \propto e^{-\beta \sum_{i=1}^{2} \omega_i \hat{n}_i} \qquad \hat{n}_i \equiv a_i^\dagger a_i \quad \hat{n} = \hat{n}_1 + \hat{n}_2$$

Let us take a bimodal system in an equilibrium state described by the density matrix presented above. To study the cooling effects we need to define a Coefficient of Performance(K) and an efficiency(η). COP is the cooling result divided by the unitless energy cost of the process, which should exist due to the second law of thermodynamics and η is the change in the photon amount of the total system divided by the sum of absolute changes in the mean photon amount of each mode.

$$K = -\frac{\Delta n_1 + \Delta n_2}{\Delta n_1 + \alpha \Delta n_2} \qquad \alpha \equiv \frac{\omega_2}{\omega_1} < 1$$

$$\eta = -\frac{\Delta n_1 + \Delta n_2}{\Delta n_1 - \Delta n_2} \qquad \eta \leq 1 - \frac{\min[\omega_1, \omega_2]}{\max[\omega_1, \omega_2]}$$

The bottom right hand side is the analogue of the **Carnot bound**.

Optimal Cooling

The minimum number of photons (optimal cooling) is represented by

$$\min_U [\, \mathrm{tr}(U \rho U^\dagger \hat{n}) \,]$$

It is very difficult to find the exact Hamiltonian to achieve the unitary dynamics for optimal cooling because it is achieved with unitaries that are permutation matrices. These are highly nonlinear from the viewpoint of $a,\ a^\dagger$. However, it is interesting to study the behavior of the system in the optimal cooling limit. Studying the permutations that achieve the optimal cooling we can find the following bound

$$-\Delta n_{\mathrm{opt}} \geq \frac{(1-y) y^{\alpha(m+1)} - (1-y^\alpha) y^m}{1 - y^{\alpha+1}} \qquad y \equiv e^{-\beta \omega_1}$$

For the optimal cooling in the limits $\omega_1 \gg \omega_2$ and $\omega_1 \simeq \omega_2$ we get these analytical results for the COP and efficiency. A numerical result of COP is also presented which showcases the fine structure of the effect.

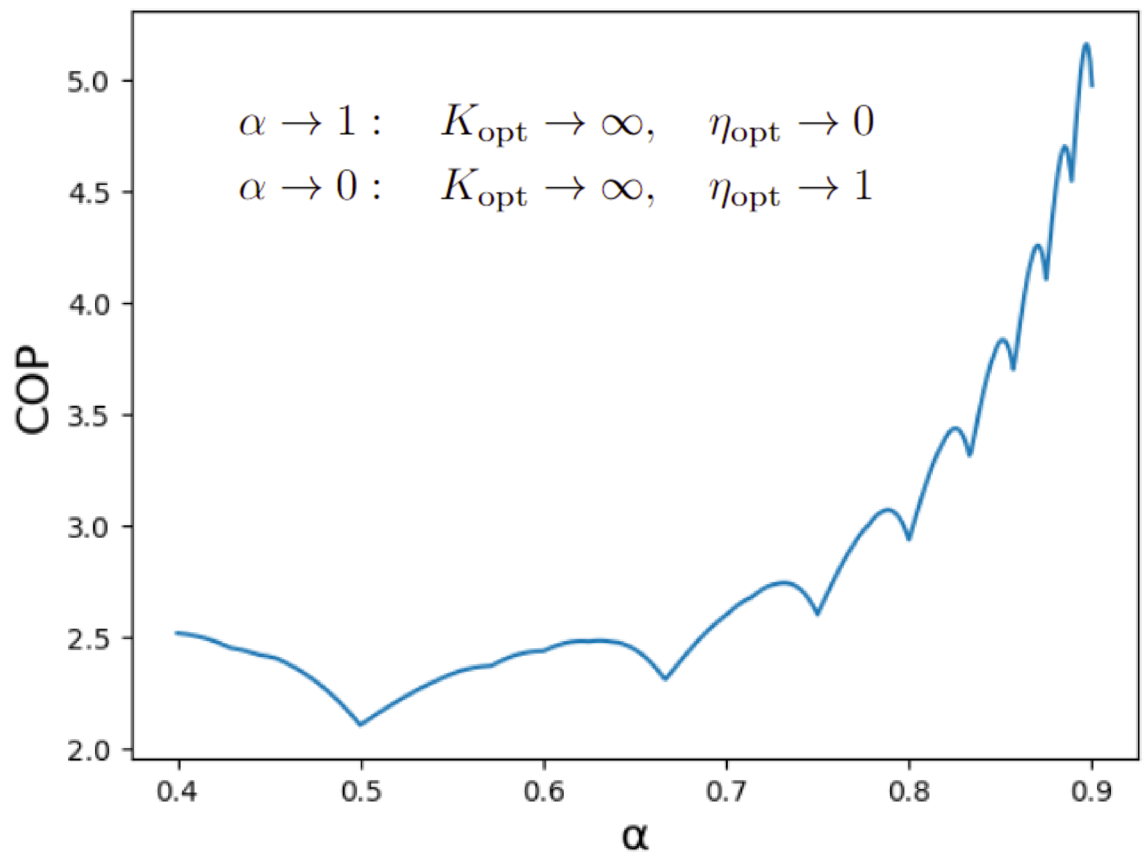


FIG. 1. The optimal coefficient of performance (COP) K_{opt} versus $\alpha = \omega_2/\omega_1 < 1$ for the optimal cooling. Here $y^\alpha = e^{-\beta\omega_2} = 0.6$ and numerical calculations are done up to the block number 300.

How the Simplest Nonlinearity Achieves Cooling

Cooling is achievable with the simplest nonlinearity in the Hamiltonian. For that we take the **cubic** Hamiltonian presented below which is achievable in anisotropic media.

$$H = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + g(a_1 + a_1^\dagger)^2 (a_2 + a_2^\dagger) + g(a_1 + a_1^\dagger)(a_2 + a_2^\dagger)^2$$

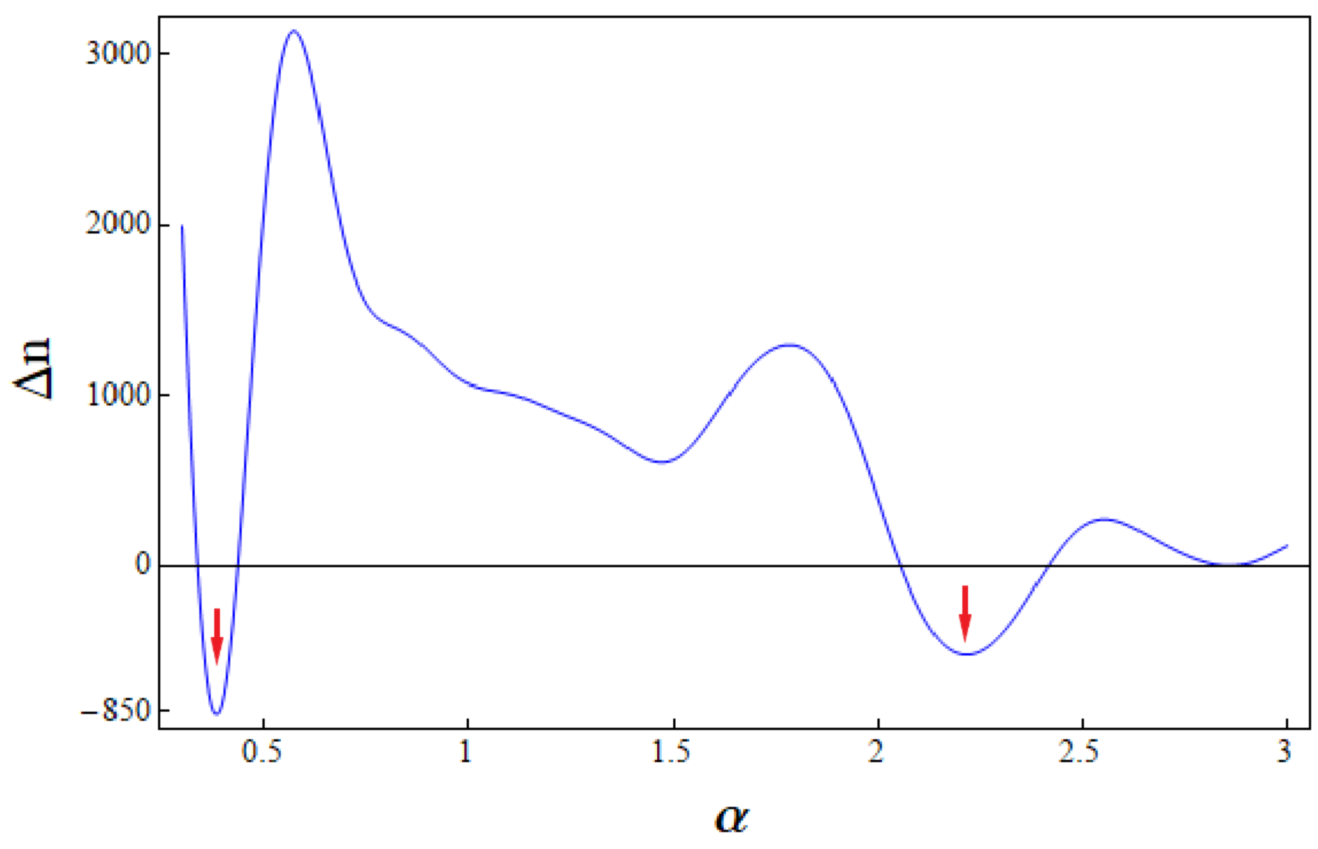


FIG. 2. The photon number difference Δn for $\omega_1 = 0.35$ and $t = 10\pi$, where $\alpha = \omega_2/\omega_1$. It is seen that near the resonating frequencies $\alpha < 0.5$ and $\alpha > 2$ the interaction Hamiltonian results in cooling. We see that $\Delta n > 0$ (no cooling) for other values of α.

As seen in Fig. 2 this Hamiltonian achieves cooling near resonances $\omega_2 \gtrsim 2\omega_1$ or $2\omega_2 \lesssim \omega_1$. Analytical results also suggest that sufficiently large cooling is only possible in these conditions. We also show that the change in the mean amount of photons near the resonating frequencies with condition $2\omega_2 \lesssim \omega_1$ is approximately equal to the change that would result from the Hamiltonian obtained using the rotating wave approximation.

$$H \simeq \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + g a_1 a_2^{\dagger 2} + g a_1^\dagger a_2^2$$

This Hamiltonian results in an **additional conservation law** presented below. It simplifies the calculation of the mean photon changes of both modes as well as the COP and efficiency.

$$2\hat{n}_1 + \hat{n}_2 = \mathrm{const} \qquad \Delta n_2 = -2\Delta n_1,\ \Delta n = -\Delta n_1$$

$$\eta = \tfrac{1}{3} \qquad K = \frac{1}{1 - 2\alpha}$$