Spins-based Quantum Otto Engine and Majorization

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Motivation

We study a quantum Otto engine with a working substance based on single spin as well as with two particles, one with a spin-1/2 and the other with an arbitrary spin-s, and coupled by Heisenberg exchange interaction, and subject to an external magnetic field. The model was studied in Ref. [1] based on the notion of heuristics whereby a worst-case scenario was analyzed to compare between two equilibrium distributions for hot and cold reservoirs, leading to positive work condition for the engine. We have characterized the operation of the quantum heat engine using the notion of majorization. The study reveals that majorization can act as a remarkable tool to analyse the performance a quantum thermal engine.

Majorization

Majorization [2] is a mathematical concept from matrix analysis, which finds applications in various areas of science ranging from mathematics, economics, social sciences, quantum information and quantum thermodynamics. It is a powerful technique for comparing two probability distributions or two density matrices in an elegant way. The concept of majorization is related to the notions of randomness and disorder and gives a preorder on probability distributions.
Suppose $P^{\downarrow} = (P_1, P_2, P_3..., P_n)$ and $P'^{\downarrow} = (P'_1, P'_2, P'_3..., P'_n)$ are two discrete probability distributions, where $P^{\downarrow}$ and $P'^{\downarrow}$ indicate that elements are taken in descending order. Then, vector P is said to be majorized by vector P', written $P \prec P'$, if the following conditions are satisfied:

$$\begin{aligned} P_n &\geq P'_n \\ P_{n-1} + P_n &\geq P'_n + P'_{n-1} \\ &\vdots \\ \sum_{k=2}^{n} P_k &\geq \sum_{k=2}^{n} P'_k \\ \sum_{k=1}^{n} P_k &= \sum_{k=1}^{n} P'_k = 1. \end{aligned}$$

Model

We consider the working substance consisting of two spins, one with spin-1/2 and other with an arbitrary spin-s, coupled by 1d isotropic Heisenberg exchange interaction in the presence of an externally applied magnetic field of magnitude B along the z-axis. The hamiltonian of the working substance is

$$H = 2B(s_z^{(1)} \otimes I^{(2)} + I^{(1)} \otimes s_z^{(2)}) + 8J(s_x^{(1)} \otimes s_x^{(2)} + s_y^{(1)} \otimes s_y^{(2)} + s_z^{(1)} \otimes s_z^{(2)}),$$

where $J \geq 0$ is the isotropic anti-ferromagnetic coupling constant. The working medium undergoes a quantum Otto cycle which consists of two quantum adiabatic and two quantum isochoric processes. In adiabatic branches magnetic field changes between two chosen values ($B_1 \to B_2 \to B_1$) at a fixed coupling strength (J), and the isochoric branches thermalise the working medium at temperature T_1 and T_2 ($T_1 > T_2$).

Majorization and Positive Work Condition

Now, only heat is exchanged between the system and reservoir during an isochoric process. The heat exchange at hot and cold reservoir is given respectively as

$$Q_1 = \sum_{k=1}^{n} \varepsilon_k (P_k - P'_k), \quad Q_2 = \sum_{k=1}^{n} \varepsilon'_k (P'_k - P_k).$$

Work done by the QOE can be written as

$$|W| = Q_1 + Q_2 = \sum_k (\varepsilon_k - \varepsilon'_k)(P_k - P'_k). \tag{1}$$

In order to illustrate our main results, we treat the case of a spin-1/2 particle coupled to a spin-1,for which $n = 6$. Then, we can evaluate

$$\begin{aligned} Q_1 = 2B_1\mathcal{X} - 12J\mathcal{Y}, \quad Q_2 = B_2\mathcal{X} - 12J\mathcal{Y}, \\ |W| = 2(B_1 - B_2)\mathcal{X}, \end{aligned} \tag{2}$$

where

$$\begin{aligned} \mathcal{X} &= 3(P_6 - P'_6) + 2(P_5 - P'_5) + 2(P_4 - P'_4) + (P_3 - P'_3) + (P_2 - P'_2), \\ \mathcal{Y} &= (P_2 - P'_2) + (P_4 - P'_4). \end{aligned}$$

For $B_1 > B_2$, PWC requires $\mathcal{X} > 0$. Now, it can be shown that under majorization $P \prec P'$, we have $\mathcal{X} \geq 0$. In this manner, majorization yields PWC for the quantum Otto cycle based on the coupled system $(1/2, 1)$.

An Upper Bound for Otto Efficiency

The efficiency of QOE $\eta = |W|/Q_1$ can be expressed as:

$$\eta = \eta_0 \left(1 - \frac{6J\mathcal{Y}}{B_1\mathcal{X}}\right)^{-1} \tag{3}$$

where $\eta_0 = 1 - B_2/B_1$ is the efficiency of uncoupled system, which is the same regardless of the magnitude of the spin.
Now, $\mathcal{X}$ is always positive when $0 \leq J \leq \Phi/6$. From the above expression of efficiency, $\mathcal{Y} > 0$ gives the regime where the Otto efficiency can be enhanced in the presence of coupling between the two spins.
Now, $\mathcal{Y} = (P_2 - P'_2) + (P_4 - P'_4) > 0$ implies $P_2 + P_4 \geq P'_2 + P'_4$. Additionally, from the majorization condition, we obtained: $P_1 \leq P'_1$. Combining these two inequalities, we get the necessary condition: $J \leq \Phi/6$, which is already the condition where majorization holds good. Thus, in this regime, we have $\mathcal{Y} \geq 0$.
Now let us consider the sign of the term $\mathcal{X} - \mathcal{Y}$. We have

$$\mathcal{X} - \mathcal{Y} = 3(P_6 - P'_6) + 2(P_5 - P'_5) + (P_4 - P'_4) + (P_3 - P'_3)$$

From the majorization conditions, we conclude: $\mathcal{X} - \mathcal{Y} \geq 0$, or $\mathcal{X} \geq \mathcal{Y}$. This implies that in the region of majorization, the efficiency has an upper bound, given by:

$$\eta_{ub} = \eta_o \left(1 - \frac{6J}{B_1}\right)^{-1} \tag{4}$$

Thus, the upper bound on the Otto efficiency is derived using the notion of majorization.

Results

We have studied the role of majorization in positive work extraction in the quantum Otto heat engine. Taking the example of spins (1/2,1) coupled system, we see the set of majorization relations implies the positive work condition. The curves for majorization and work extraction show that whenever majorization holds, work is always positive.

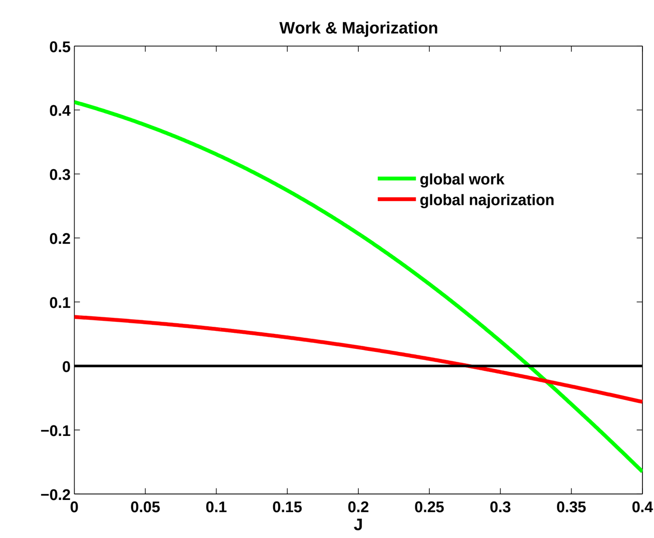

Fig. 1: $B_1 = 5, B_2 = 3, T_1 = 6, T_2 = 3$

Conclusions

A heuristic-based approach and using techniques of worst-case/best-case reasoning, the regime in which the machine yields positive work has been shown [1]. Here, we have analyzed the performance of a quantum Otto engine based on a working medium with two coupled spins using the theory of majorization. We find that when the canonical probability distribution at the hot bath is majorized by the canonical probability at cold bath, then majorization relations gives positive work condition. It is an interesting problem to extend this analysis to more general working medium.

References

[1]. R. S. Johal and V. Mehta, Quantum heat engines with complex working media, complete Otto cycles and heuristics, Entropy vol. 23, 1149 (2021).
[2]. A. W. Marshall, I. Olkin, and B. C. Arnold, Inequalities: Theory of majorization and its applications, Springer Series in Statistics, Springer New York, NY (1980).