Optimal local work extraction from bipartite quantum systems in the presence of Hamiltonian couplings

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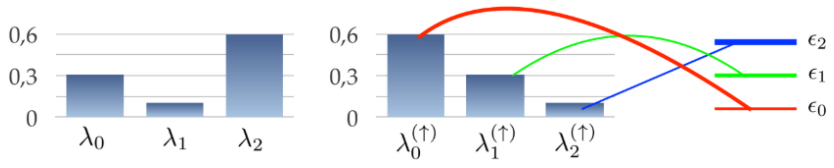
SCUOLA NORMALE SUPERIORE

Ergotropy

The *ergotropy* of a system is the **maximum work** that can be extracted from it with unitary operations

$$\mathcal{E}(\rho, H) = \mathrm{Tr}[\rho H] - \min_U \mathrm{Tr}[U\rho U^\dagger H]$$



It is a convex and non-linear functional of ρ and H

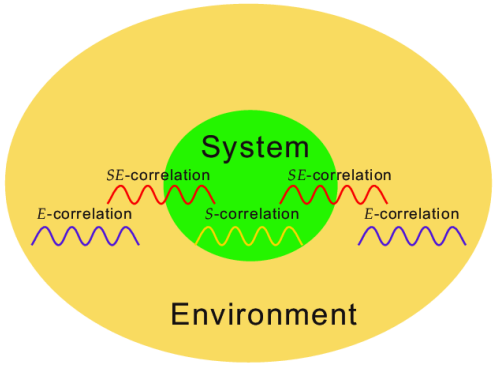
$\mathcal{E}(\hat{\rho}) = 0$ for passive states

$\mathcal{E}(\hat{\rho}) = E(\hat{\rho})$ if and only if the state is pure

Local ergotropy

We define the *local ergotropy* as the maximum extractable work with **local unitary operations**:

$$\mathcal{E}_S(\rho_{SE}, H_{SE}) = \mathrm{Tr}[\rho_{SE} H_{SE}] - \min_{U_S} \mathrm{Tr}[(U_S \otimes I_E)\rho_{SE}(U_S^\dagger \otimes I_E)H_{SE}]$$

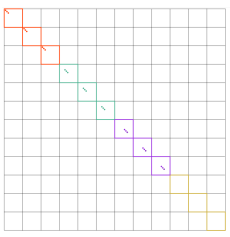


References:
- Ergotropy:
Allahverdjan et al. EPL 67 (2004)
- Passive states:
Lenard JSP 19 (1978), Alicki JPA (1979)
- Extraction of work from local systems:
Mukherjee et al. Pyhs. Rev. E 93 (2016)
Alhambra et al. PRL 123 (2019)

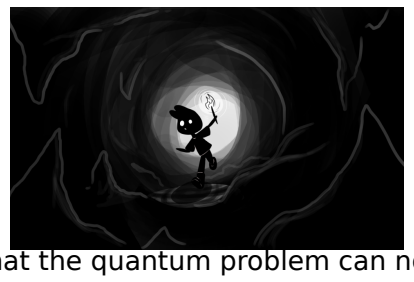
A difficult problem...

In the classical case (UNITARIES → PERMUTATIONS), the problem reduces to finding, given a probability distribution p_{ij} over energies ϵ_{ij} ,

$$\min_\pi \sum_{i,j} p_{\pi(i),j}\epsilon_{i,j}$$



This is the **assignment problem**, which can be efficiently solved with an algorithm. But no closed formula exists for its solution.



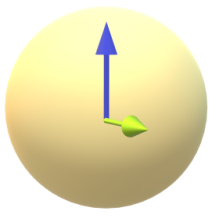
Hence we know that the quantum problem can not, in general, be solved with a closed formula.

...but it can be solved for qubits...

We derive an exact closed formula for the local ergotropy in the special case in which the local system S is a **two-level system**:

$$\mathcal{E}_S(\hat{\rho}_{SE}, \hat{H}_{SE}) = \begin{cases} \mathrm{Tr}[|\mathcal{M}| - \mathcal{M}] , & \text{for } \det[\mathcal{M}] \geq 0, \\ \mathrm{Tr}[|\mathcal{M}| - \mathcal{M}] - \frac{2}{\|\mathcal{M}^{-1}\|} , & \text{for } \det[\mathcal{M}] < 0, \end{cases}$$

where *M* is a 3x3 matrix which depends in a complicated way on the state and on the Hamiltonian, and which is calculated using the formalism of **Generalized Pauli Operators**



...or relaxed to a SDP

The convexity of the local ergotropy functional allows us to relax the problem from a maximum over local unitaries to a maximum over **convex conbination** of local unitaries. If we relax further to local **unital** transformations, with the Choi isomorphism we can formulate a Semidefinite Programming bound:

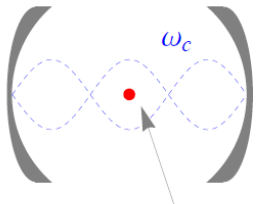
$$\mathcal{E}_S(\hat{\rho}_{SE}, \hat{H}_{SE}) \leq \mathrm{Tr}\left[\hat{H}_{SE}\hat{\rho}_{SE}\right] - \min_{E_{SS'}} \mathrm{Tr}\left[C_{SS'}E_{SS'}\right] ,$$

$$\text{with} \qquad \begin{aligned} &E_{SS'} \geq 0 ; \\ &\mathrm{Tr}_S\, E_{SS'} = \mathbb{I}_{S'} ; \\ &\mathrm{Tr}_{S'}\, E_{SS'} = \mathbb{I}_S , \end{aligned}$$

Example: atom in a cavity

Jaynes-Cummings Hamiltonian:

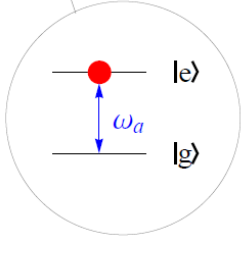
$$\hat{H}_{SE}^{(\mathrm{JC})} := \hbar\omega_E\, \hat{a}^\dagger\hat{a} + \frac{\hbar\omega_S}{2}\hat{\sigma}_{S,3} + \frac{\hbar\Omega}{2}\left(\hat{a}\otimes\hat{\sigma}_S^+ + \hat{a}^\dagger\otimes\hat{\sigma}_S^-\right)$$



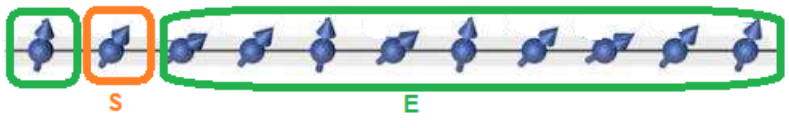
We find that the local ergotropy of the *dressed eigenstates* of an atom in a cavity is always bigger than the ergotropy of the isolated atom (taking into account the energetic cost of switching the interaction off).

Local ergotropy of the $|n,+\rangle$ eigenstate:

$$\mathcal{E}_S(|n,+\rangle, \hat{H}_{SE}^{(\mathrm{JC})}) = \hbar\omega_S\cos 2\theta_n + \frac{\sqrt{n+1}}{2}\hbar|\Omega\sin 2\theta_n| ,$$



Example: XXZ spin chain



$$\mathcal{E}_S(\hat{\rho}_{SE}\,; \hat{H}_{SE}) = \begin{cases} 0 & (|k| \leq N/4) \\ -\frac{8J}{N}\cos\left(\frac{2\pi k}{N}\right) & (|k| > N/4) \end{cases}$$

The local ergotropy of a spin site in a XXZ chain can be nonzero even if the local state is passive (i.e. would have zero ergotropy)