Geometric Bounds on the Power of Adiabatic Thermal Machines

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Introduction

Is there a fundamental bound on the power of microscopic thermal devices that operate close to Carnot efficiency?
We show that the power of microscopic thermal machines, operating between two fixed temperatures, **decays quadratically** as Carnot efficiency is approached.

- This result presents a **geometric trade-off** between power and efficiency.
- The result is **universal** for devices with any kind of thermodynamically consistent dynamics, for both **refrigerators** and **heat engines.**

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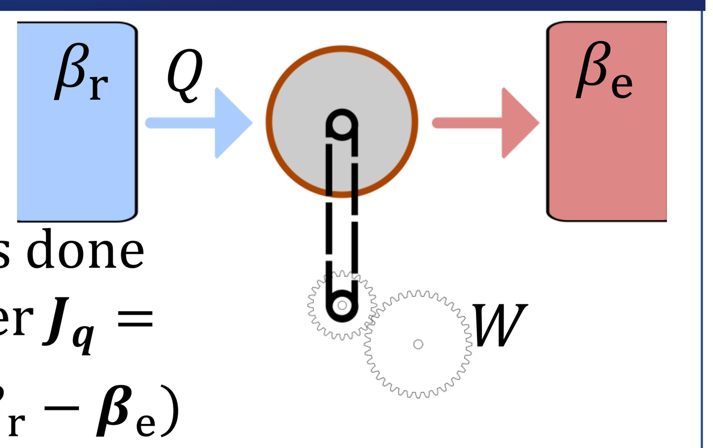
Microscopic Thermal Machines

We consider devices that operate between two heat baths $\boldsymbol{\beta}_\mathrm{r}$ and $\boldsymbol{\beta}_\mathrm{e}$.



- **Refrigerator** ($\boldsymbol{\beta}_\mathrm{r} > \boldsymbol{\beta}_\mathrm{e}$): work $\boldsymbol{W \equiv J_w}$, is done during cycle period $\boldsymbol{\tau}$ with cooling power $\boldsymbol{J_q = Q/\tau}$ and efficiency $\boldsymbol{\varepsilon = \frac{Q}{W} \leq \varepsilon_C \equiv \beta_\mathrm{e}/(\beta_\mathrm{r} - \beta_\mathrm{e})}$
- **Heat Engines** ($\boldsymbol{\beta}_\mathrm{r} < \boldsymbol{\beta}_\mathrm{e}$): work $\boldsymbol{W \equiv J_w}$, is produced during cycle period $\boldsymbol{\tau}$, uptake $\boldsymbol{J_q = Q}$ and efficiency $\boldsymbol{\eta = \frac{W}{Q} \leq \eta_C = (\beta_\mathrm{r} - \beta_\mathrm{e})/\beta_\mathrm{r}}$

Adiabatic Response

The isothermal work, J_w^{iso}, and quasi-static heat current J_q^{qs} are defined.

$$J_w^{\mathrm{iso}} = J_w\Big|_{A_q=0} = K_{ww}A_w \qquad J_q^{\mathrm{qs}} = J_q\Big|_{A_w=0} = K_{qq}A_q$$

Work input and heat flux,

$$J_w - J_w^{\mathrm{iso}} = K_{wq}A_q. \qquad J_q - J_q^{\mathrm{qs}} = K_{qw}A_w.$$

- Affinities $A_q = \beta_\mathrm{e} - \beta_\mathrm{r}$ and $A_w = \beta_\mathrm{e}/\tau$.
- Onsager symmetry, $K_{qw}|_{A_x\to 0} = -K_{wq}|_{A_x \to 0}$, is assumed.

The efficiency,
$$\frac{\varepsilon}{\varepsilon_\mathrm{C}} = -\frac{K_{qw} + K_{qq}\left(\frac{A_q}{A_w}\right)}{K_{wq} + K_{ww}\left(\frac{A_w}{A_q}\right)}$$

As $\tau \to \infty$, Carnot efficiency can only be reached if the **quasi-static heat current**, K_{qq}, vanishes and if $\boldsymbol{A_q \propto A_w^\alpha}$ where $0 < \alpha < 1$.

Given $\boldsymbol{A_q \propto A_w^\alpha}$, where $0 < \alpha < 1$, we expand the off diagonal coefficients in A_q,

$$K_{qw} \to L_{qw} + L_{qw}^q A_q \qquad K_{wq} \to L_{wq} + L_{wq}^q A_q$$

The work and heat current in **adiabatic response** (AR),

$$J_q = L_{qw}A_w + L_{qw}^q A_w A_q \qquad J_w = L_{ww}A_w - L_{qw}A_q + L_{wq}^q A_q^2$$

Thermodynamic Geometry

Thermodynamic quantities,

$$J_w = -\int_0^\tau dt\, f_t^\mu \dot{\lambda}_t^\mu, \qquad J_q = \frac{1}{\tau}\int_0^\tau dt\, j_t$$

Thermodynamic force f_t^μ, **instantaneous heat-current** j_t.
Set of control parameters $\boldsymbol{\lambda}_t$.

f_t^μ and j_t are expanded,

- $f_t^\mu \simeq -\partial_\mu \mathcal{F}_{\lambda_t} - \beta_\mathrm{e}\mathcal{R}_{\lambda_t}^{\mu\nu}\dot{\lambda}_t^\nu + \mathcal{R}_{\lambda_t}^{\mu q}A_q + \mathcal{R}_{\lambda_t}^{\mu qq}A_q^2,$
- $j_t \simeq \beta_\mathrm{e}\mathcal{R}_{\lambda_t}^{q\mu}\dot{\lambda}_t^\mu + \beta_\mathrm{e}\mathcal{R}_{\lambda_t}^{qq\mu}\dot{\lambda}_t^\mu A_q$

$\mathcal{R}_\lambda$ and $\mathcal{F}_\lambda$ depend only on λ and β_e.

The AR coefficients have a **geometric interpretation**,

$$\begin{bmatrix} L_{wq} & L_{qw} \\ L_{wq}^q & L_{qw}^q \end{bmatrix} = \oint_\gamma \begin{bmatrix} \mathcal{A}_\lambda^{\mu q} & \mathcal{A}_\lambda^{q\mu} \\ \mathcal{A}_\lambda^{\mu qq} & \mathcal{A}_\lambda^{qq\mu} \end{bmatrix} d\lambda^\mu.$$

Thermodynamic potentials $\mathcal{A}_\lambda^{xyz}$ depend on coefficients $\mathcal{R}_{\lambda_t}^{xyz}$. $\boldsymbol{\gamma}$ is path taken in the space of control parameters.

Approaching Carnot Efficiency

The efficiency under new scaling argument,
$$\frac{\varepsilon}{\varepsilon_\mathrm{C}} = 1 + \frac{L_{qw}^q + L_{wq}^q}{L_{qw}}A_q + \frac{L_{ww}}{L_{qw}}\frac{A_w}{A_q}$$

maximised with respect to A_q,

$$\frac{\varepsilon}{\varepsilon_\mathrm{C}} \leq 1 - \sqrt{A_w L_{qw}/Z} \qquad \text{saturated if} \quad A_q^* = -\sqrt{zA_w}$$

with $Z = L_{qw}^3/4\,(L_{qw}^q + L_{wq}^q)L_{ww}$ and $z = L_{ww}/(L_{qw}^q + L_{wq}^q)$

Main Result: Power-Efficiency trade-off

For refrigerators (heat engines), cooling power P_c (power P),

$$P_c \leq \mathcal{Z}(\varepsilon_\mathrm{C} - \varepsilon)^2/\varepsilon_\mathrm{C}^2 \qquad P \leq \mathcal{Z}(\eta_\mathrm{C} - \eta)^2/\eta_\mathrm{C}$$

- Power decays **quadratically** as Carnot approached.
- The **bound is geometric**. The figure of merit Z becomes geometric $Z \to \mathcal{Z}$, substituting the geometric AR coefficients.

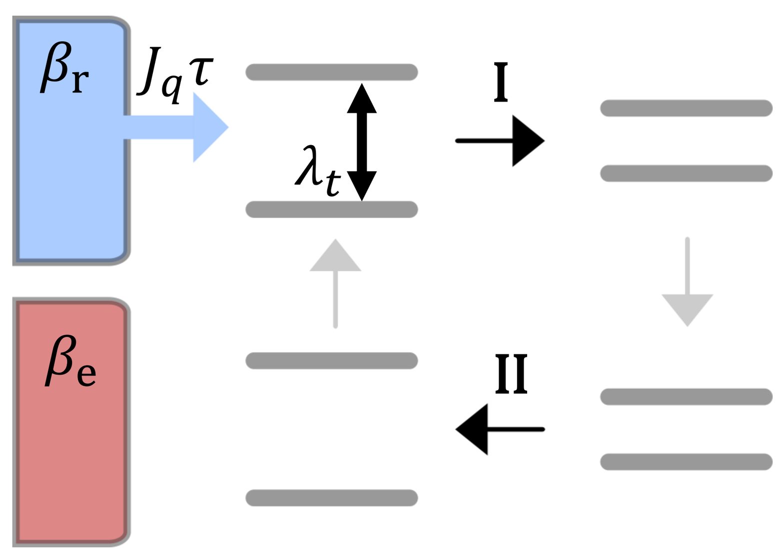
Example: Two-Stroke Refrigerator

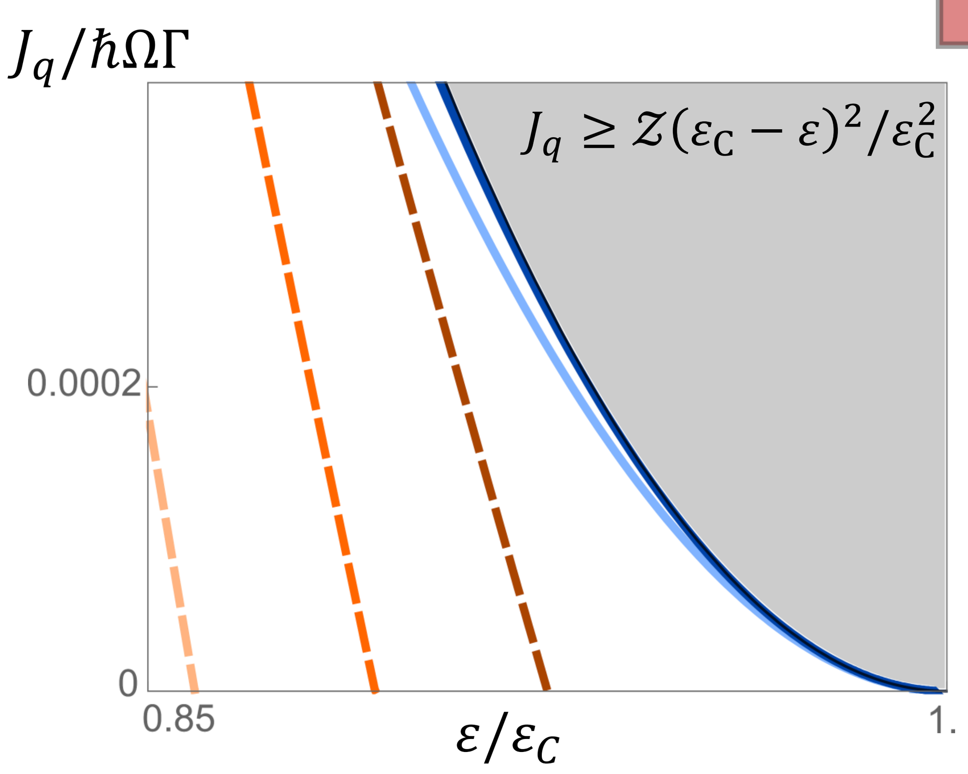
Single-qubit refrigerator: **Hamiltonian,** $H_\lambda = \hbar\Omega\lambda_t\sigma_z/2$ in a two-stroke cycle.

$\Gamma > 0$ sets relaxation timescale of system.

Stroke I: couple to β_r, $\lambda_0 \to \lambda_1$.

Stroke II: couple to β_e, $\lambda_1 \to \lambda_\tau \equiv \lambda_0$.



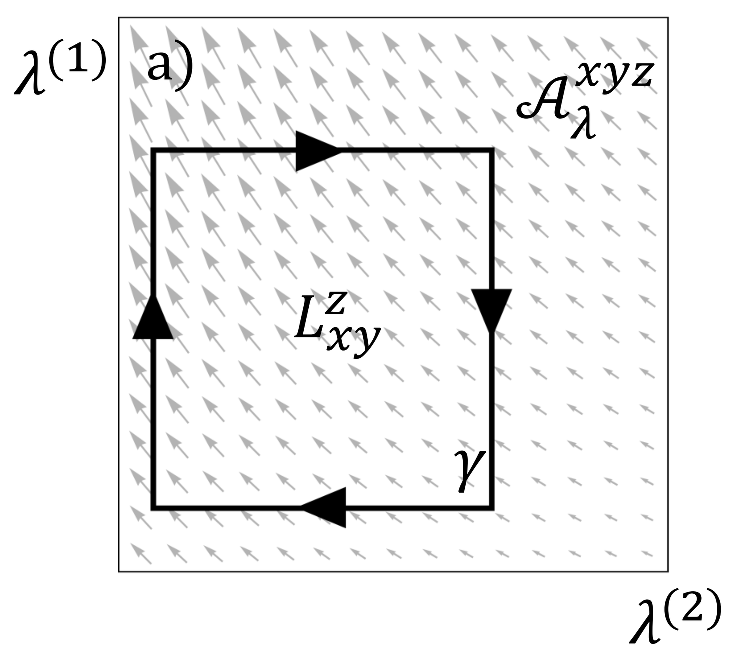


- Fixed A_q (dashed-lines)
- Unobtainable region (grey) is **geometrically bounded**.
- Bound saturated by **optimal scaling $\boldsymbol{A_q^*}$** (light-blue) and **optimal driving** (dark-blue)

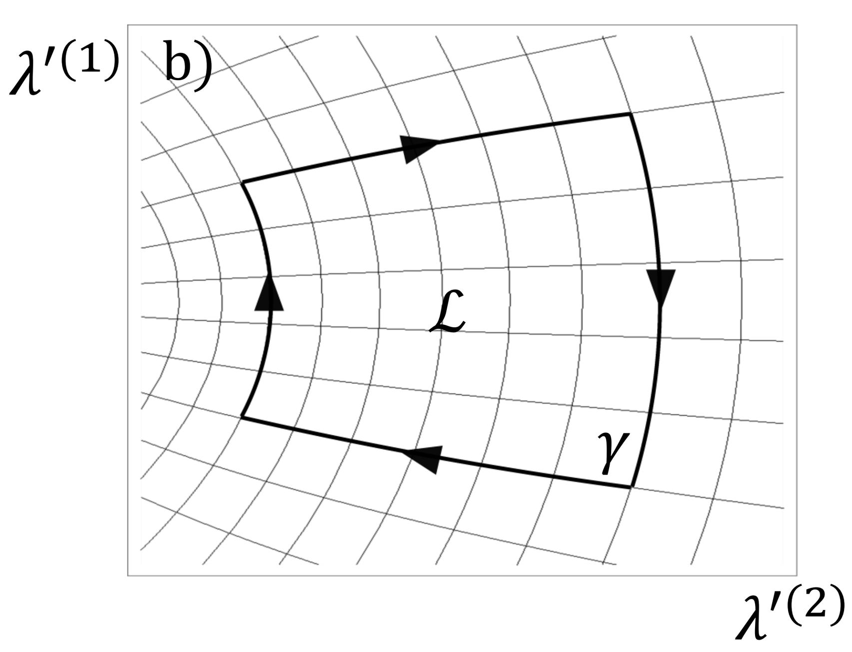
Coefficient L_{ww} has **geometric bound** corresponding to the **thermodynamic length $\boldsymbol{\mathcal{L}}$** characterised by metric $\mathcal{G}_\lambda^{\mu\nu}$.

$$L_{ww} \geq \mathcal{L}^2 \quad \text{where} \quad \mathcal{L} \leq \oint_\gamma \sqrt{\mathcal{G}_\lambda^{\mu\nu}d\lambda^\mu d\lambda^\nu}$$

The bound is **saturated** by driving λ_t by the **optimal speed function** ϕ_t



a) Thermodynamic vector potential



b) Thermodynamic length in curvilinear coordinates $\lambda \to \lambda'$