Stochastic entropy production for restricted quantum state diffusion

Jonathan Dexter and Ian Ford

Department of Physics and Astronomy, University College London, Gower St, London WC1E 6BT, U.K.

UCL

The message

- Stochastic entropy production is associated with the stochastic dynamics of an open system.
- Diffusion increases the uncertainty in a system coordinate and this creates entropy.
- There follows a rather technical but important issue!
- What if the diffusion matrix has null-eigenvectors with vanishing eigenvalues?
- These identify non-diffusive system coordinates that do not participate in entropy production.
- Thus 'dynamical' and 'spectator' coordinates. Only the former contribute entropy.
- The procedure for computing stochastic entropy production from the stochastic equations of motion assumes that all variables are dynamical.
- We therefore need to recast the procedure. We demonstrate this for an open quantum system.

Dynamics and Thermodynamics

- Stochastic dynamics:

$$d\boldsymbol{x} = \left(\boldsymbol{A}^{\text{rev}} + \boldsymbol{A}^{\text{ir}}\right) dt + \boldsymbol{B}d\boldsymbol{W}$$

- Environmental stochastic entropy production [1]:

$$\begin{aligned} d\Delta s_{\text{env}} = \sum_{i,j}\Bigg\{ & \frac{D^{-1}_{ij}(\boldsymbol{x})}{2}\left(A^{\text{ir}}_i(\boldsymbol{x})dx_j + A^{\text{ir}}_j(\boldsymbol{x})dx_i\right) \\ & - \frac{D^{-1}_{ij}(\boldsymbol{x})}{2}\left(\left(\sum_n \frac{\partial D_{jn}(\boldsymbol{x})}{\partial x_n}\right)dx_i + \left(\sum_m \frac{\partial D_{im}(\boldsymbol{x})}{\partial x_m}\right)dx_j\right) \\ & + \frac{1}{2}\sum_k\Bigg[D_{ik}(\boldsymbol{x})\frac{\partial}{\partial x_k}\left(D^{-1}_{ij}(\boldsymbol{x})A^{\text{ir}}_j(\boldsymbol{x})\right) + D_{jk}(\boldsymbol{x})\frac{\partial}{\partial x_k}\left(D^{-1}_{ij}(\boldsymbol{x})A^{\text{ir}}_i(\boldsymbol{x})\right) \\ & - D_{ik}(\boldsymbol{x})\frac{\partial}{x_k}\left(D^{-1}_{ij}(\boldsymbol{x})\left(\sum_n \frac{\partial D_{jn}(\boldsymbol{x})}{\partial x_n}\right)\right) - D_{jk}(\boldsymbol{x})\frac{\partial}{x_k}\left(D^{-1}_{ij}(\boldsymbol{x})\left(\sum_m \frac{\partial D_{im}(\boldsymbol{x})}{\partial x_m}\right)\right)\Bigg]dt\Bigg\} \end{aligned}$$

$$\boldsymbol{D} = \frac{1}{2}\boldsymbol{B}\boldsymbol{B}^T$$

- What happens if the diffusion matrix $\boldsymbol{D}$ is singular?

Example in quantum state diffusion [2,3]

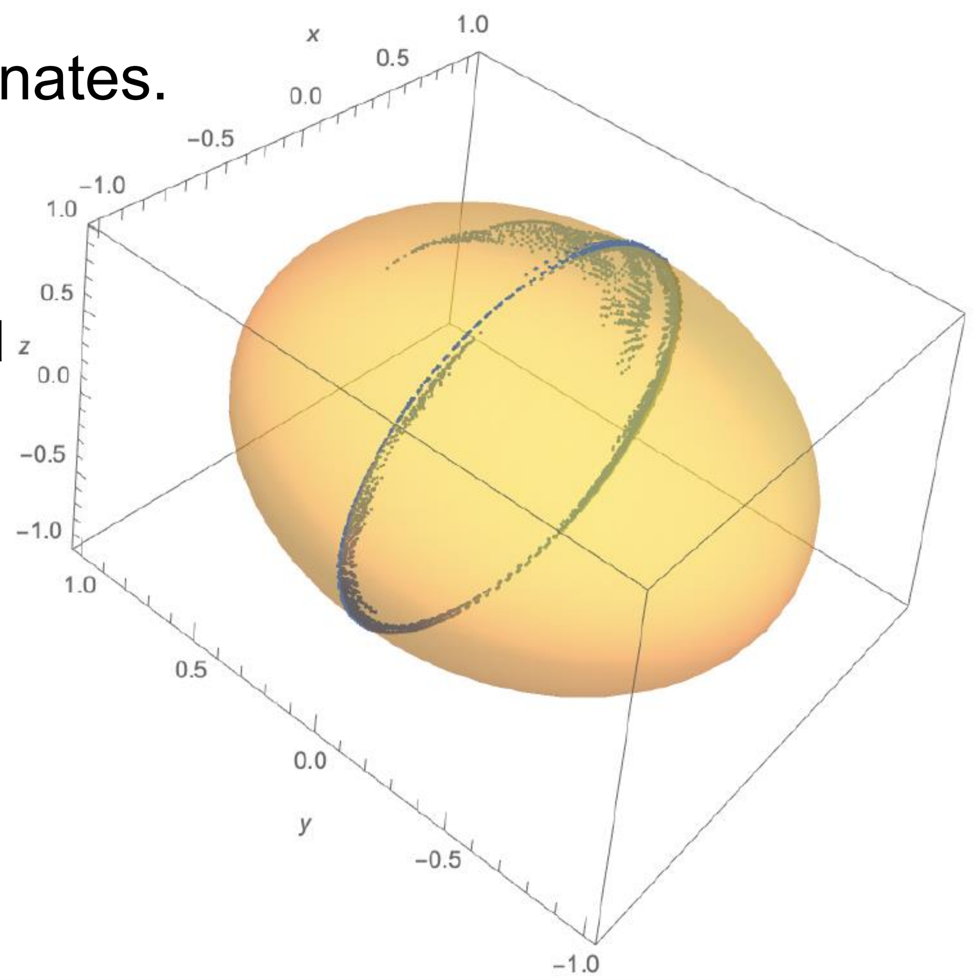
Two level system coupled to an environment through raising and lowering operators. Singular $\boldsymbol{D}$ matrix:

$$\boldsymbol{D} = \begin{pmatrix} x^4 - 2x^2 + z^2 + 1 & x\left(x^2-1\right)y & x\left(x^2-2\right)z \\ x\left(x^2-1\right)y & x^2y^2 & x^2yz \\ x\left(x^2-2\right)z & x^2yz & x^2\left(z^2+1\right) \end{pmatrix}$$

in Bloch sphere coordinates.

Coordinates on the ellipsoid are diffusive and entropy producing. '**Dynamical**'.

Coordinate normal to the ellipsoid is in this case an invariant. '**Spectator**'.



Easy to resolve in this case

- New dynamical coordinates θ, ϕ :

$$x = \sin(\theta)\cos(\phi) \qquad z = \cos(\theta)$$

$$y = b\sin(\theta)\sin(\phi)$$

- Equations of motion:

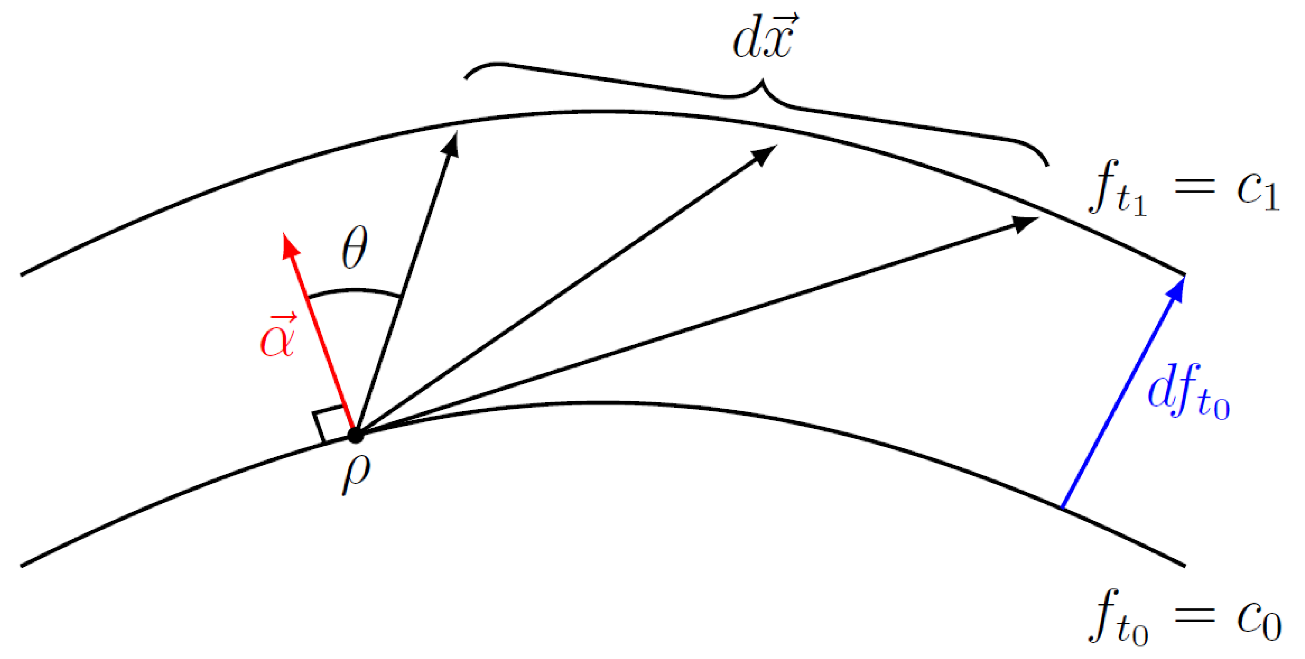
$$d\theta = A_1 dt + B_{11}dW_1 + B_{12}dW_2$$

$$d\phi = A_2 dt + B_{21}dW_1 + B_{22}dW_2$$

- 2×2 $\boldsymbol{D}$ matrix in θ, ϕ is non-singular.

How can we proceed more generally?

The equations of motion do not always possess invariants: we instead seek coordinate transformations that identify **deterministic** spectator variables $f(\vec{x}(t))$ evolving as $df = G(\vec{x})dt$.



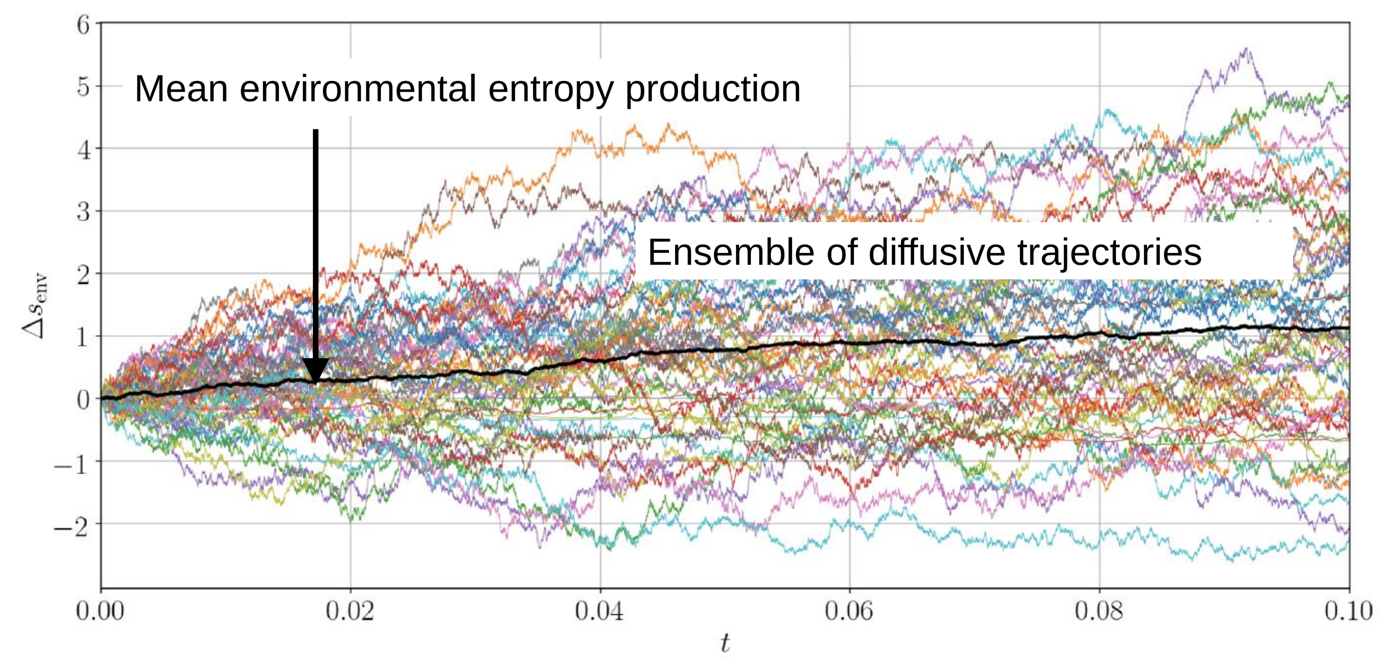
Contours of the spectator variables lie normal to the null eigenvectors $\vec{\alpha}$ of the $\boldsymbol{D}$ matrix.

$d\vec{x}$ consists of a deterministic increment in spectator variable f plus a stochastic increment along the contour $f = c_1$, specified here by the angle θ.

Dynamics are diffusive in the coordinates normal to $\vec{\alpha}$, the null-eigenvectors of the diffusion matrix.

Stochastic entropy production

Inverse and derivatives of the non-singular $\boldsymbol{D}$ matrix are used to compute the stochastic entropy production [4].



References:

[1] R.E. Spinney and I.J. Ford. Entropy production in full phase space for continuous stochastic dynamics, Physical Review E, 85 051113 (2012).
[2] C.L. Clarke, Irreversibility measures in a quantum setting, PhD thesis, UCL 2021.
[3] J. Dexter, A measurement free approach to entropy production and irreversibilityin open quantum systems: MSc thesis, UCL 2021.
[4] I.J. Ford, Zeno effects and entropy production for Brownian trajectories of a physical density matrix. Oral presentation at this conference