Thermodynamic uncertainty relation in degenerate and nondegenerate maser heat engines

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Abstract

In this work, we investigate the thermodynamic uncertainty relation, which represents a trade-off between entropy production rate and relative power fluctuations, for non-degenerate three-level and degenerate four-level maser heat engines. For the non-degenerate case, we study two slightly different configurations of three-level maser engine and compare degree of violation of thermodynamic uncertainty relation in both models. We also show that the thermodynamic uncertainty relation remains invariant when we scale the matter-field coupling constant and system-bath coupling constants by the same factor. Further, for the degenerate four-level engine, we study the effects of noise-induced coherence on the thermodynamic uncertainty relation. We show that depending on the parametric regime of operation, the phenomenon of noise-induced coherence can either enhance or suppress the relative power fluctuations.

Thermodynamic uncertainty relation (TUR)

- **For steady-state classical heat engines obeying the Markovian dynamics, TUR represents a trade-off between the rate of entropy production (σ) and relative fluctuations in the power of the engine [1]:**

$$\mathcal{Q} \equiv \frac{\sigma\, \text{var}(P)}{P^2} \geq 2, \tag{1}$$

  where var(P) and P represent variance of the power and average power, respectively.
- In the presence of coherence in quantum systems, TUR can be violated. In the following, we study the role of quantum coherence, coherence induced by a semiclassical driving as well as noise-induced coherence, on the violation of TUR in a maser heat engine.

Model

- **Model I**

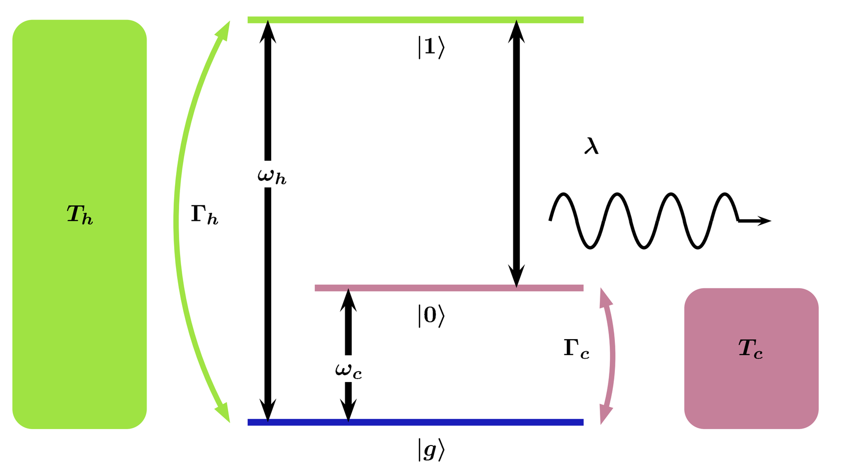

Figure 1: Model of three-level laser heat engine [2] continuously coupled to two reservoirs of temperatures T_h and T_c having coupling constants Γ_h and Γ_c, respectively. The system is interacting with a classical single mode field. λ represents the strength of matter-field coupling.

- **Hamiltonian of the system and Interaction Term**

$$H_0 = \hbar \sum \omega_k |k\rangle\langle k|, \quad V(t) = \hbar\lambda(e^{-i\omega t}|1\rangle\langle 0| + e^{i\omega t}|0\rangle\langle 1|).$$

- **Time evolution of the system in a rotating frame**

$$\dot{\rho} = -\frac{i}{\hbar}[V_R, \rho] + \mathcal{L}_h[\rho] + \mathcal{L}_c[\rho], \tag{2}$$

where $V_R = \hbar\lambda(|1\rangle\langle 0| + |0\rangle\langle 1|)$. $\mathcal{L}_{h(c)}$ represents the dissipative Lindblad superoperator describing the system-bath interaction with the hot (cold) reservoir:

$$\begin{aligned}\mathcal{L}_h[\rho] &= \Gamma_h(n_h+1)\big(\sigma_{g1}\rho\sigma_{g1}^\dagger - \frac{1}{2}\{\sigma_{g1}^\dagger\sigma_{g1}, \rho\}\big) \\ &+ \Gamma_h n_h\big(\sigma_{g1}^\dagger\rho\sigma_{g1} - \frac{1}{2}\{\sigma_{g1}\sigma_{g1}^\dagger, \rho\}\big),\end{aligned} \tag{3}$$

$$\begin{aligned}\mathcal{L}_c[\rho] &= \Gamma_c(n_c+1)\big(\sigma_{g0}\rho\sigma_{g0}^\dagger - \frac{1}{2}\{\sigma_{g0}^\dagger\sigma_{g0}, \rho\}\big) \\ &+ \Gamma_c n_c\big(\sigma_{g0}^\dagger\rho\sigma_{g0} - \frac{1}{2}\{\sigma_{g0}\sigma_{g0}^\dagger, \rho\}\big),\end{aligned} \tag{4}$$

where $\sigma_{gk} = |g\rangle\langle k|$, $k = 0, 1$.

- Here $n_{h(c)} = 1/(\exp[\hbar\omega_{h(c)}/k_B T_{h(c)}] - 1)$ is average occupation number of photons in hot (cold) reservoir satisfying the relations $\omega_c = \omega_0 - \omega_g$, $\omega_h = \omega_1 - \omega_g$.

Power and efficiency

- **For a weak system-bath coupling, the output power, the heat flux and the efficiency of the engine can be defined [3], as follows:**

$$P = \frac{i}{\hbar}\text{Tr}([H_0, V_R]\rho_R), \quad \dot{Q}_h = \text{Tr}(\mathcal{L}_h[\rho_R]H_0), \quad \eta = \frac{P}{\dot{Q}_h}. \tag{5}$$

Comparison of TUR for two slightly different models of SSD engine

- Here, we introduce another model of three-level maser heat engine slightly different from the Model I.

- **Model II**

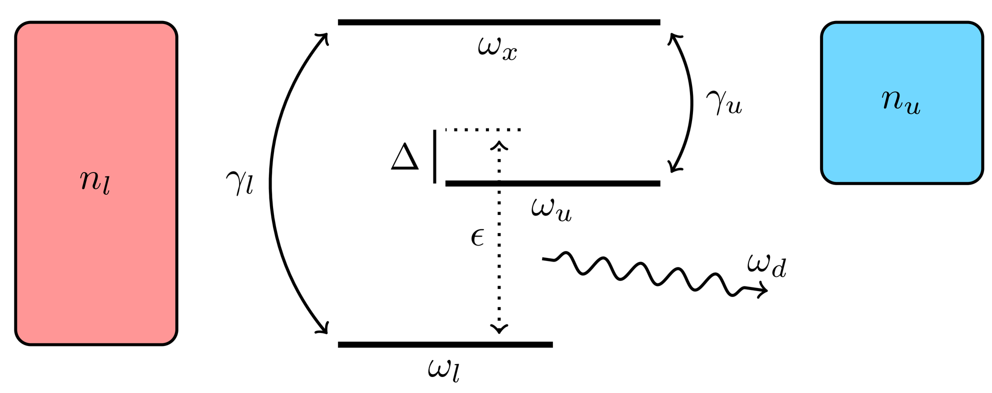

Figure 2: Model II of three-level maser heat engine.

- Model II is slightly different from the Model I. Here, cold reservoir is connected to two upper levels instead of two lower levels as in Model I. Similarly, power mechanism is coupled to lower two levels instead of upper two levels.
- Although both models look similar, the resulting TUR relations are different. The difference can be traced back to the dynamical equations of motion, which are different for the abovesaid models.

Method: Full counting statistics

- By dressing the Lindblad superoperator [4], either $\mathcal{L}_h[\rho]$ or $\mathcal{L}_c[\rho]$, by a counting field χ and then vectorizing the density matrix elements into a state vector $\rho_R = (\rho_{gg}, \rho_{00}, \rho_{11}, \rho_{10}, \rho_{01})^T$, we can write the Lindblad master equation as a matrix equation with the Liouvillian supermatrix $\mathbf{L}(\chi)$

$$\dot{\rho}_R = \mathbf{L}(\chi)\rho_R. \tag{6}$$

- In the steady-state limit, the cumulant generating function is given by

$$G(\chi) = -\epsilon_0(\chi), \tag{7}$$

  where ϵ_0 is is the ground-state energy (or the eigenvalue of the smallest real part) of the superoperator $\mathbf{L}(\chi)$.
- The CGF supplies all cumulants, specifically the steady-state average power and variance of the power:

$$P = -\frac{\partial\epsilon_0(\chi)}{\partial(i\chi)}\Big|_{\chi=0}, \qquad \text{var}(P) = -\frac{\partial^2\epsilon_0(\chi)}{\partial(i\chi)^2}\Big|_{\chi=0}. \tag{8}$$

- Using the abovesaid method, we can find TUR quantifier $\mathcal{Q}$ in both models of SSD engine. Although we obtain different expressions for $\mathcal{Q}_1$ and $\mathcal{Q}_2$, in both cases, $\mathcal{Q}$ is function of Γ_h, Γ_c, λ, n_h and n_c:

$$\mathcal{Q}_{1,2} \equiv \mathcal{Q}_{1,2}(\Gamma_h, \Gamma_c, \lambda, n_h, n_c)) \tag{9}$$

Comparison between two models

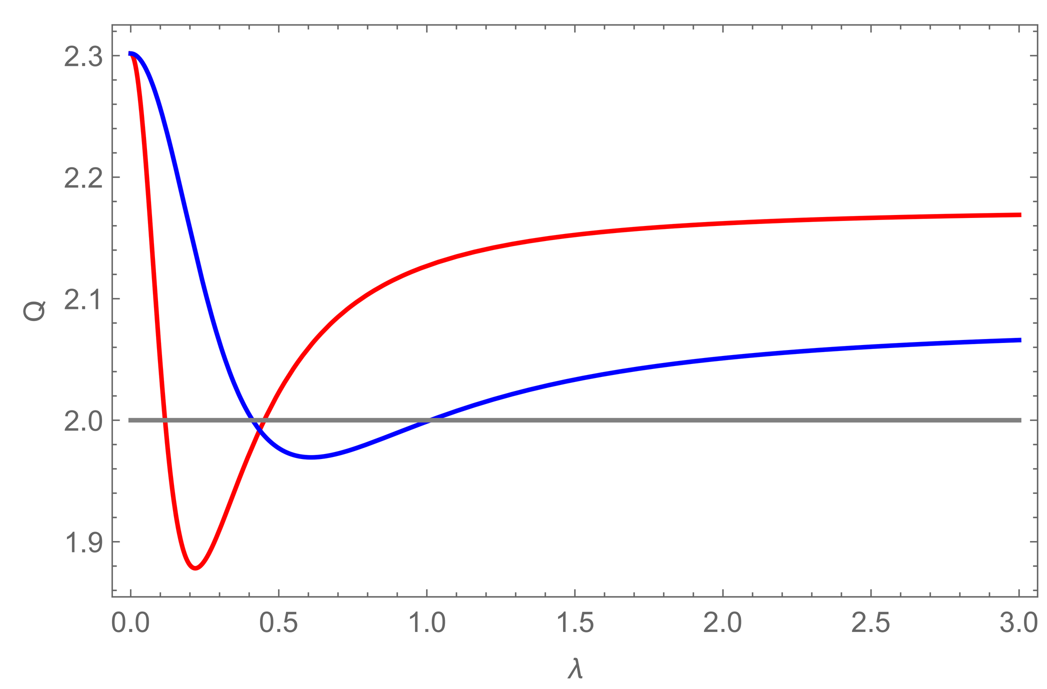

Figure 3: TUR quantifier $\mathcal{Q}$ versus matter-field coupling parameter λ. Blue and red curves correspond to Model I and Model II, respectively. Here, $\Gamma_h = 0.1$, $\Gamma_c = 2$, $n_h = 5$, $n_c = 0.027$.

- **Realiability of the engine**

$$\mathcal{R} = \frac{P}{\sqrt{var(P)}}. \tag{10}$$

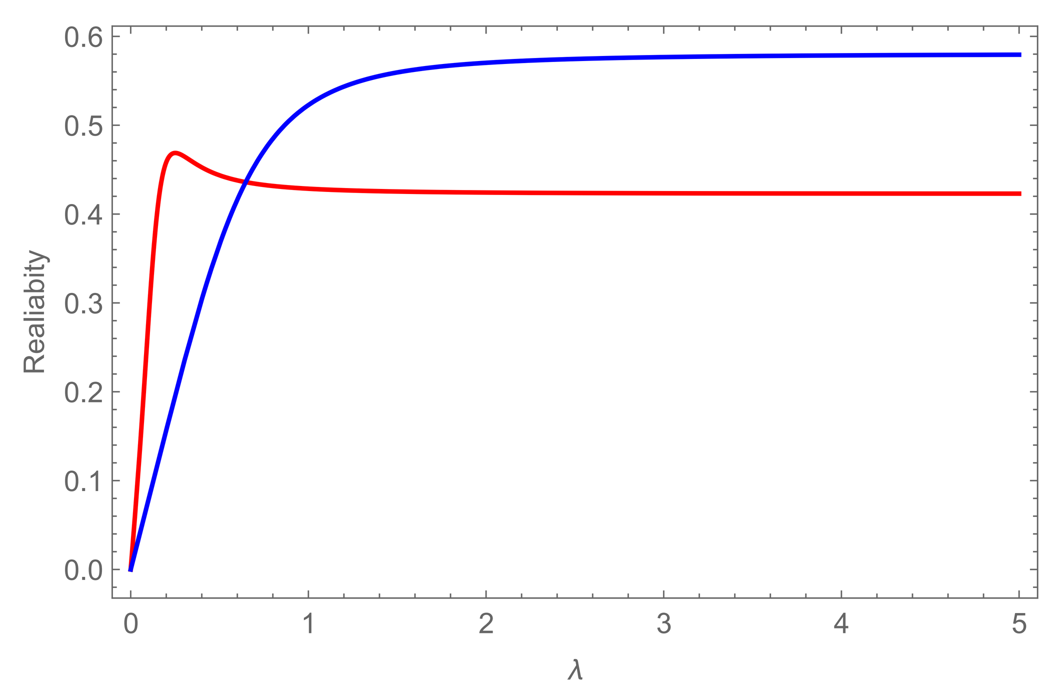

Figure 4: Realiability $\mathcal{R}$ versus matter-field coupling parameter λ. Blue and red curves correspond to Model I and Model II, respectively. Here, $\Gamma_h = 0.1$, $\Gamma_c = 2$, $n_h = 5$, $n_c = 0.027$.

A few interesting observations

- **Scaling property**

$$\mathcal{Q}(\Gamma_h, \Gamma_c, \lambda, n_h, n_c) = \mathcal{Q}(k\Gamma_h, k\Gamma_c, k\lambda, n_h, n_c) \tag{11}$$

- **In the high-temperature limit, both models obey the same TUR relation**

$$\mathcal{Q}_{HT} = 2 - \frac{16(n_h - n_c)^2\Gamma_h\Gamma_c\lambda^2\big(\Gamma_c^2 n_c^2 + \Gamma_h^2 n_h^2 + 5\Gamma_c\Gamma_h n_h n_c + \lambda^2\big)}{9n_h n_c(\Gamma_c n_c + \Gamma_h n_h)^2(4\lambda^2 + \Gamma_h\Gamma_c n_h n_c)^2}. \tag{12}$$

  In the high temperature regime, we can ignore spontaneous emission as compared to the stimulated emission, which makes both models equivalent as they yield same dynamical equations.
- **In the high-temperature limit, $\mathcal{Q}_{HT}$ is always smaller than 2.**

$$\mathcal{Q}_{HT} < 2 \tag{13}$$

  This is very interesting result as $\mathcal{Q}_{HT}$ always violates classical TUR relation.

Noise-induced coherence

- To study the effects of noise-induced coherence on TUR, we consider a four-level maser heat engine model with degeneracy in upper states.

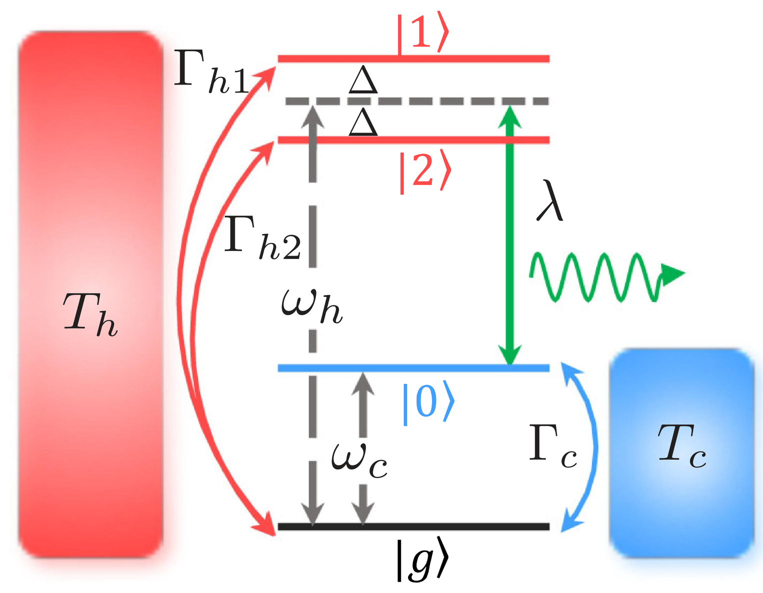

Figure 5: Four-level maser heat engine.

- The phenomenon of noise-induced coherence arises due to interference between two decaying channels ($|1\rangle \to |g\rangle$ and $|2\rangle \to |g\rangle$ transitions) to the same level $|g\rangle$.
- The strength of NIC is determined by

$$p = \cos\theta = \frac{d_{g1}.d_{g2}}{|d_{g1}|\,|d_{g2}|}. \quad [d_{gk} = \langle g|\mathbf{d}|k\rangle].$$

Effect of noise-induced coherence on TUR

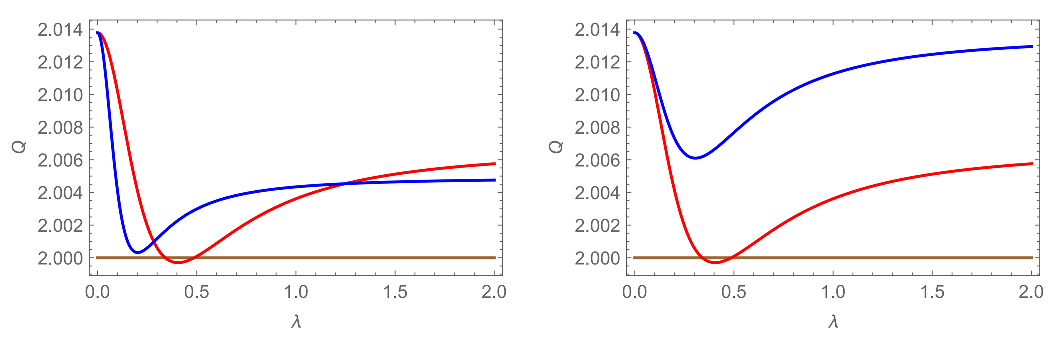

Figure 6: TUR quantifier $\mathcal{Q}$ as a function of λ. Red and blue curves represent TURs for the three-level model (Model I) and four-level model with noise-induced coherence, respectively. Here, $n_h = 2$, $n_c = 1$, $\Gamma_h = 0.2$, $\Gamma_c = 0.1$. $p = -0.6$ and $p = 0.6$ for the left and right panel, respectively.

Conclusions

- Classical TUR relation can be violated in the three-level maser heat engine.
- Spontaneous emission plays important role in the degree of violation of classical TUR as in the absent of spontaneous emission, Model I and Model II discussed here yield the same TUR.
- In the high-temperature limit, classical TUR is always violated.
- Depending on the parametric regime of operation, the phenomenon of noise-induced coherence can either enhance or suppress the relative power fluctuations.

References

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