# Quantum Trajectory Framework for General Time-Local Master Equations

Brecht Donvil[1] and Paolo Muratore-Ginanneschi[2]

[1]Institute for Complex Quantum Systems and IQST, Ulm University - Albert-Einstein-Allee 11, D-89069 Ulm, Germany
[2]University of Helsinki, Department of Mathematics and Statistics P.O. Box 68 FIN-00014, Helsinki, Finland

## Introduction

Master equations are one of the main avenues to study open quantum systems. When the master equation is of the Lindblad form, its solution can be "unraveled in quantum trajectories".
We show that general time-local, trace-preserving master equations can also be **unravelled in terms of an ordinary quantum trajectories**. The crucial ingredient is to weigh averages by a $1d$ scalar which we call the **"influence martingale"**. We thus extend the existing theory **without increasing the computational complexity**.
The influence martingale allows us to define a pseudo-probability measure for the quantum trajectories. We define a proper time reversal and use it to derive a **fluctuation relation**.

## General Time Local Mast Equations

We are interested in time local master equations of the form

$$\dot\rho(t) = -i\,[H(t)\,,\rho(t)] + \sum_k \Gamma_k(t)\left(L_k\,\rho(t)L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho(t)\}\right)$$

where **the weights $\Gamma_k(t)$ have no positivity requirement**. We assume that the Lindblad operators satisfy the completeness relation $\sum_{k=1}^N L_k^\dagger L_k = \mathbb{I}$. We can represent the dynamics in terms of a normal stochastic Schrödinger equation:

$$\begin{cases} d\psi(t) = & -iH(t)\psi(t)dt - \frac{1}{2}\sum_l \gamma_k(t)(L_k^\dagger L_k - \|L_k\psi(t)\|^2)\psi(t)dt \\ & +\sum_k\left(\frac{L_k\psi(t)}{\|L_k\psi(t)\|} - \psi(t)\right)dN_k(t) \\ d\mu(t) = & C(t)\mu(t)dt + \mu(t)\sum_k\left(\frac{\Gamma_k(t)}{\gamma_k(t)} - 1\right)dN_k(t) \end{cases}$$

$$\mathbb{E}(dN_k(t)|\psi) = \gamma_k(t)\|L_k\psi(t)\|^2 dt$$
$$\gamma_k(t) > 0, \qquad \gamma_k(t) - \Gamma_k(t) = C(t) > 0 \quad \forall\, k$$

The state is reconstructed by $\rho(t) = \mathbb{E}(\mu(t)\psi(t)\psi^\dagger(t))$

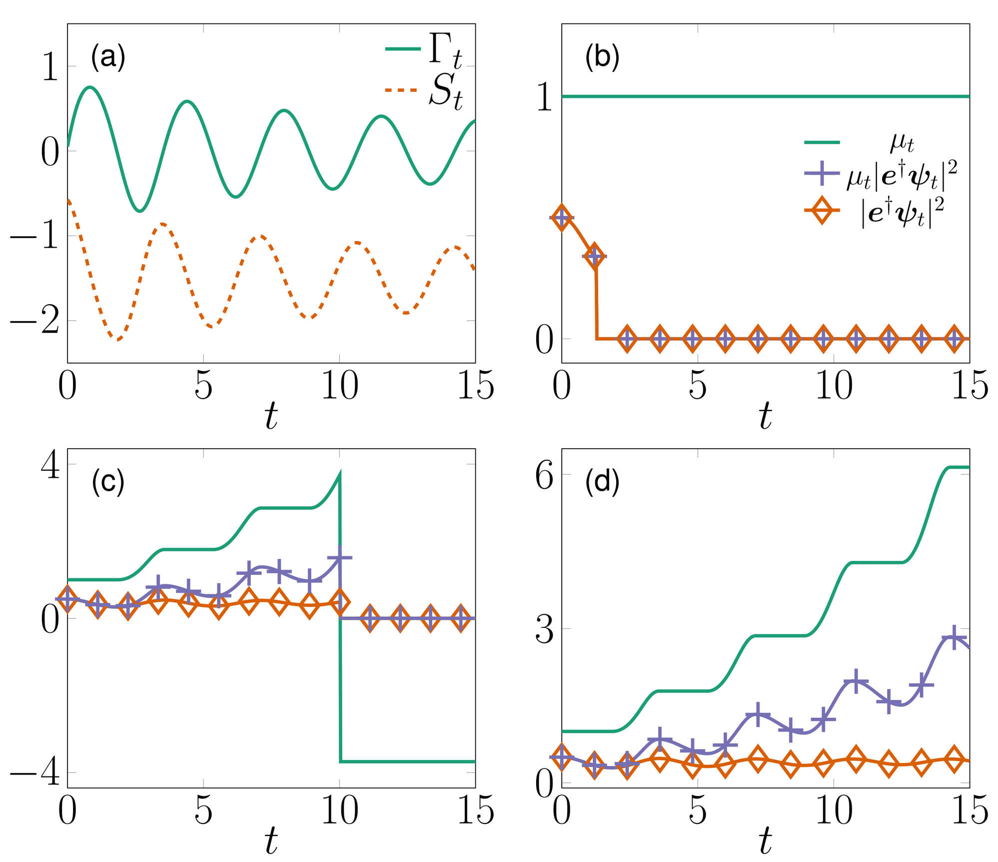


Fig. 1: Illustration of the influence martingale for a qubit master equation

## Performance for Large Quantum Systems

We study a qubit chain described by the Hamiltonian

$$H = \sum_{k=1}^{L}\sigma_+^{(k)}\sigma_-^{(k)} + \lambda\sum_{k=1}^{N-1}(\sigma_+^{(k)}\sigma_-^{(k+1)} + \sigma_+^{(k+1)}\sigma_-^{(k)}).$$

The first qubit is connected to a channel with a non-positive weight, the chain dynamics are governed by the master equation

$$\dot\rho(t) = -i\,[H\,,\rho(t)] + \Gamma(t)\left(\sigma_-^{(1)}\rho(t)\sigma_+^{(1)} - \frac{1}{2}\{\sigma_+^{(1)}\sigma_-^{(1)}, \rho(t)\}\right)$$
$$+\gamma\sum_{k=2}^{L}\left(\sigma_-^{(k)}\rho(t)\sigma_+^{(k)} - \frac{1}{2}\{\sigma_+^{(k)}\sigma_-^{(k)}, \rho(t)\}\right) + \delta\sum_{k=1}^{L}\left(\sigma_+^{(k)}\rho(t)\sigma_-^{(k)} - \frac{1}{2}\{\sigma_-^{(k)}\sigma_+^{(k)}, \rho(t)\}\right)$$

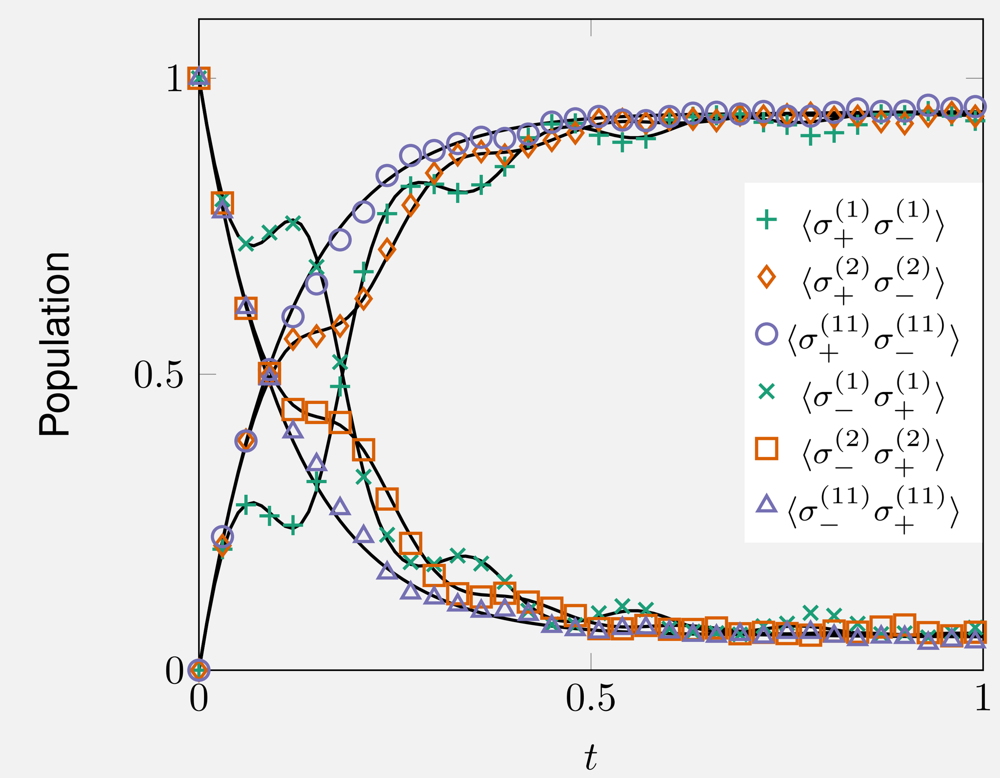


Fig. 2: Population of sites 1, 4 and 11 for the qubit chain with $L = 11$. The marks show the result of the influence martingale and the full black lines the result of numerically integrating the master equation. $\Gamma(t) = \gamma - 12\exp(-2t^3)\sin^2(15t)$, $\delta = (1/0.129)0.063$, $\gamma = (1/0.129)(1 + 0.063)$.

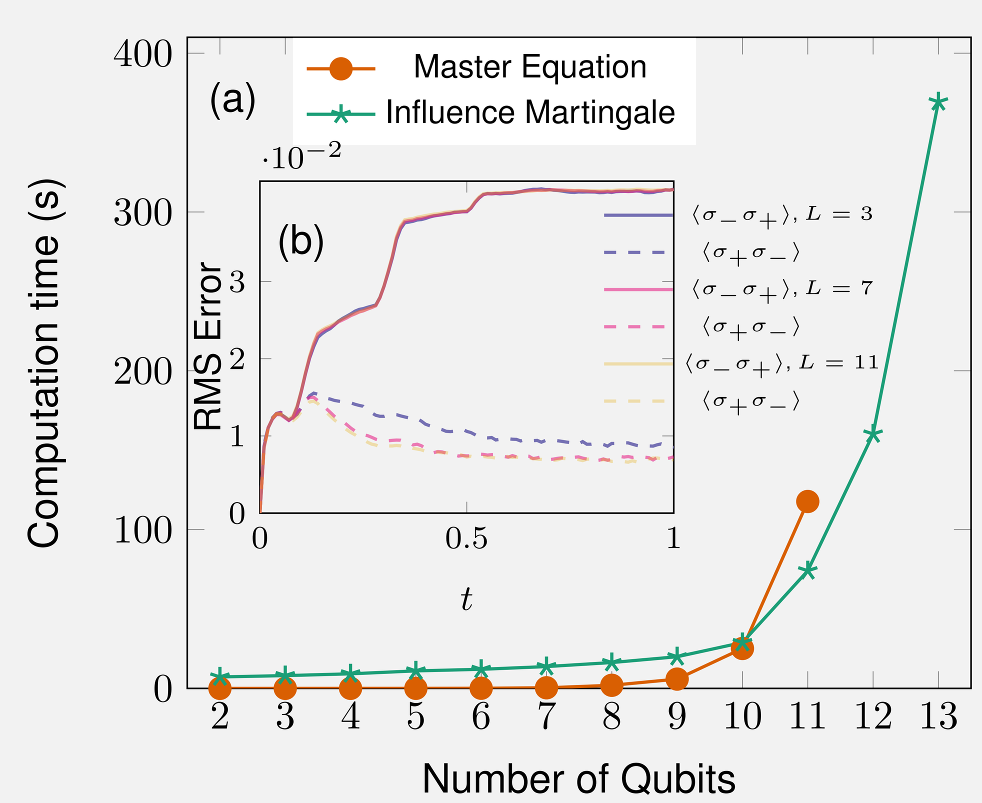


Fig. 3: (a) Computation time for both the Master Equation and Influence Martingale method as a function of the amount of qubits in the chain. For the stochastic method we generated 1000 realizations. (b) The root mean square error of the populations averaged over all individual sites.

## Fluctuation Relation

Let $\{L_{k_j}, t_j\}$ be a quantum trajectory with jumps described by the operators $L_{k_j}$ at times $t_j$. The pseudo probability of the trajectory with an initial state $\phi_i$ (measured with probability $p_i$) and final state measured to be $\phi_f$, is given by

$$P(\{L_{k_j}, t_j\}_{j=1}^N, \phi_f, \phi_i) = e^{\int_0^\tau C(t)dt}\left(\prod_{j=1}^N \Gamma_{k_j,t_j}\right) p_i\,|\langle\phi_f|U_{\tau,t_N}L_{k_j}\dots U_{t_1,0}|\phi_i\rangle|^2.$$

We define the jump operators for the time reversed operators by [2]

$$\bar L_k = L_k^\dagger$$
$$\bar\Gamma_k(t) = a_k(\tau - t)\Gamma_k(\tau - t)$$

where $a_{k,t} > 0$. The time reversal is then well defined when the condition

$$\sum_k \Gamma_k(t)(L_k^\dagger L_k - a_k(t)L_kL_k^\dagger) = 0$$

holds.

Comparing the probability $P$ of the forward and $P_R$ of the backward trajectories, we find

$$\frac{P_R(\{L_{k_{N+1-j}}^\dagger, \tau - t_j\}_{j=1}^N, \phi_i, \phi_f)}{P(\{L_{k_j}, t_j\}_{j=1}^N, \phi_f, \phi_i)} = \mathcal{A} = \frac{p_f}{p_i}\prod_{j=0}^N a_{k_j}(t_j)$$

We therefore have the fluctuation relation $\langle\mathcal{A}\rangle = 1$.

## Conclusion

- General time local master equations can be unravelled in terms of ordinary stochastic Schrödinger equations. We have shown that this allows for efficient integration of master equations for large systems.
- The influence martingale allows to define pseudo-probability measures for quantum trajectories. In this way, we can extend the methods from Lindblad master equations to derive a fluctuation relation.

## References

1 B. Donvil and P. Muratore-Ginanneschi, Nat. Commun. (accepted), arXiv:2102.10355 (2021)

2 G. Manzano, J. M. Horowitz, and J. M. Parrondo, Phys. Rev. X 8, 031037 (2018).