Precision bound in periodically modulated continuous quantum thermal machines (arXiv: 2204.14005)

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Motivation

Along with average thermodynamic quantities of interest, fluctuations about the mean values are important for determining the quality of output for quantum technologies. Very recently it has been shown that, for arbitrary out of equilibrium scenario, there is a trade-off between precision (fluctuations) and the cost (entropy production rate), which is broadly known as Thermodynamic Uncertainty Relations (TUR). TUR provides a deep understanding in analyzing the precision of thermal machines, first shown for steady state classical continuous heat engine. Further generalizations include both classical and quantum scenarios like autonomous and periodic quantum thermal machines, information driven engine and thermoelectric junctions. We study a periodically driven continuous quantum thermal machine, which is to our knowledge still unexplored.

Counting field statistics

Total Hamiltonian of the system (WM) and two connected baths for the machine is,

$$H(t) = H_S(t) + H_B + H_{SB} = H_0(t) + H_{SB}, \tag{1}$$

where, $H_B = H_h + H_c$ and, $H_{SB} = H_{Sh} + H_{Sc} = \mathcal{S} \otimes \mathcal{B}_h + \mathcal{S} \otimes \mathcal{B}_c$. Initial state: $\rho(0) = \rho_S(0) \otimes \rho_B$, where $\rho_B = \rho_h \otimes \rho_c$, prepared in thermal states. We introduce $\chi \equiv \{\chi_h, \chi_c\}$ to denote collectively both the counting variables. Generating function for the TPM scheme,

$$\mathcal{G}(\chi, t) = \mathrm{Tr}_{SB}[\rho(\chi, t)] = \mathrm{Tr}[\rho_S(\chi, t)], \tag{2}$$

$$\rho(\chi, t) = U(\chi, t)\rho(0)U^\dagger(-\chi, t), \quad U(\chi, t) = e^{-i(\chi_h H_h + \chi_c H_c)/2} U(t) e^{i(\chi_h H_h + \chi_c H_c)/2}. \tag{3}$$

The generating function allows us to evaluate the statistics of energy transferred between system and each reservoir: $\langle \Delta E_j^n \rangle = \frac{\partial^n}{\partial (i\chi_j)^n} \mathcal{G}(\chi, t)\,|_{\chi=0}$.

Minimal thermal machine model

$$H_S(t) = H_S(t+T); \quad H_S(t) = \frac{1}{2}\omega(t)\sigma_z, \tag{4}$$

where period $T = 2\pi/\Delta$ [1]. Additionally, we consider $\mathcal{S} = \sigma_x$. One can represent $\tilde{\mathcal{S}}(t) = \tilde{\sigma}_x(t) = U_S^\dagger(t)\sigma_x U_S(t)$ in the Floquet basis as,

$$\tilde{\sigma}_x(t) = \sum_{q \in \mathcal{Z}} (\eta(q) e^{-i(\omega_0 + q\Delta)t} \sigma^- + \eta^*(q) e^{i(\omega_0 + q\Delta)t} \sigma^+), \tag{5}$$

$$\eta(q) = \frac{1}{T} \int_0^T \exp\left(i \int_0^t (\omega(s) - \omega_0)\, ds \right) e^{-iq\Delta t} dt. \tag{6}$$

Here, the Floquet basis are just the eigenvectors of σ_z. We perform a secular approx (neglecting the terms with $q \neq q'$), and get,

$$\partial_t \tilde{\rho}_S(\chi, t) = \sum_{q, j = \{h,c\}} \mathcal{L}_i^q(\chi_j, t)[\tilde{\rho}_S(\chi, t)] =$$

$$\sum_{q, j=\{h,c\}} \frac{P_q}{2} \Big(G_j(\omega_0 + q\Delta)[2e^{-i(\omega_0+q\Delta)\chi_j} \sigma^- \tilde{\rho}_S(\chi, t)\sigma^+ - \sigma^+\sigma^- \tilde{\rho}_S(\chi, t) - \tilde{\rho}_S(\chi, t)\sigma^+\sigma^-]$$

$$+ G_j(-\omega_0 - q\Delta)[2e^{i(\omega_0+q\Delta)\chi_j} \sigma^+ \tilde{\rho}_S(\chi, t)\sigma^- - \sigma^-\sigma^+ \tilde{\rho}_S(\chi, t) - \tilde{\rho}_S(\chi, t)\sigma^-\sigma^+] \Big). \tag{7}$$

Generating function

For the above master equation, the evolution of the diagonal and off-diagonal elements (in Floquet basis) are decoupled. So, it is enough to consider the evolution of the diagonal entries of $\tilde{\rho}_S(\chi, t)$, as the generating function is given by the trace of $\tilde{\rho}_S(\chi, t)$.

$$\begin{pmatrix} \dot{\tilde{\rho}}_{00}(\chi, t) \\ \dot{\tilde{\rho}}_{11}(\chi, t) \end{pmatrix} = \mathcal{L}(\chi) \begin{pmatrix} \tilde{\rho}_{00}(\chi, t) \\ \tilde{\rho}_{11}(\chi, t) \end{pmatrix} \equiv \begin{bmatrix} l_{00} & l_{01}^\chi \\ l_{10}^\chi & l_{11} \end{bmatrix} \begin{pmatrix} \tilde{\rho}_{00}(\chi, t) \\ \tilde{\rho}_{11}(\chi, t) \end{pmatrix}, \tag{8}$$

We define the cumulant generating function as, $\mathcal{C}(\chi, t) \equiv \log \mathcal{G} = \log \mathrm{Tr}[\tilde{\rho}_S(\chi, t)]$. The steady state is reached in the limit of long times, when the cumulant generating function is dominated by the eigenvalue $\lambda(\chi)$ of $\mathcal{L}(\chi)$ with the largest real part.

$$\lim_{t\to\infty} \mathcal{C}(\chi, t) \approx \lambda(\chi)t, \quad \lambda(\chi) = \frac{1}{2}(l_{00} + l_{11}) + \frac{1}{2}\sqrt{(l_{00} + l_{11})^2 - 4(l_{00}l_{11} - l_{01}^\chi l_{10}^\chi)}. \tag{9}$$

Currents and fluctuations

$$\langle J_j \rangle = \lim_{t\to\infty} \frac{d}{dt} \frac{\partial}{\partial (i\chi_j)} \mathcal{C}(\chi, t)\Big|_{\chi=0} = \sum_q \frac{P_q(\omega_0 + q\Delta)}{w+1} G_j(\omega_0 + q\Delta)[e^{-\beta_j(\omega_0+q\Delta)} - w]. \tag{10}$$

$$\frac{p_1^{ss}}{p_2^{ss}} \equiv w = \frac{\sum_{q,j} P_q G_j(\omega_0 + q\Delta) e^{-\beta_j(\omega_0+q\Delta)}}{\sum_{q,j} P_q G_j(\omega_0 + q\Delta)} = \frac{l_{11}}{l_{00}}. \tag{11}$$

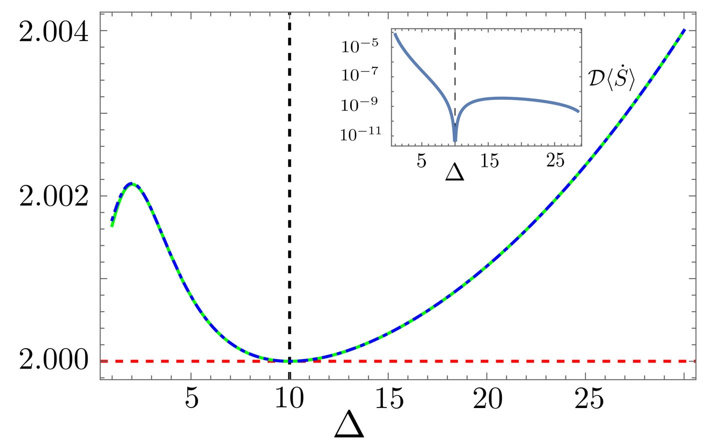
Here we used the KMS condition, $G_j(-\omega_0 - q\Delta) = e^{-\beta_j(\omega_0+q\Delta)} G_j(\omega_0 + q\Delta)$. Similarly the current fluctuation is given as,

$$\mathrm{var}(J_j) = \lim_{t\to\infty} \frac{d}{dt} \frac{\partial^2}{\partial (i\chi_j)^2} \mathcal{C}(\chi, t)\Big|_{\chi=0} = \frac{\partial^2 \lambda(\chi)}{\partial (i\chi_j)^2}\Big|_{\chi=0}. \tag{12}$$

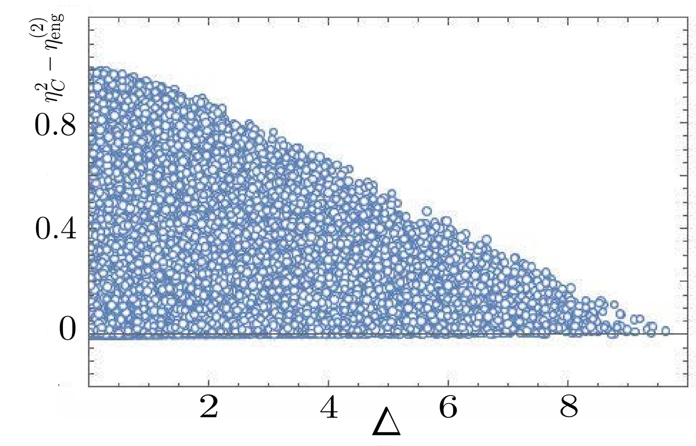
$$\langle \dot{S} \rangle = -\beta_h \langle J_h \rangle - \beta_c \langle J_c \rangle, \quad \langle \mathcal{P} \rangle = -\langle J_h \rangle - \langle J_c \rangle. \tag{13}$$

Sinusoidal modulations and TUR

We focus on the sinusoidal modulation: $\omega(t) = \omega_0 + \lambda\,\Delta\,\sin(\Delta t)$, with $0 \le \lambda \le 1$. This allows us to consider only the harmonics $q = 0, \pm 1$, with, $P_0 \approx 1 - \frac{\lambda^2}{2}$, and $P_{\pm 1} \approx \frac{\lambda^2}{4}$, with the higher order harmonics $P_q \approx 0$ for $|q| > 1$ in the limit of small λ. Additionally, we consider the bath spectral functions, $G_c(\omega) \approx 0$ for $\omega \ge \omega_0$, and $G_h(\omega) \approx 0$ for $\omega \le \omega_0$. The machine works as a heat engine for $\Delta < \Delta_{cr}$ and as a refrigerator for $\Delta > \Delta_{cr}$, where $\Delta_{cr} = \omega_0 (T_h - T_c) / (T_h + T_c)$. The power output vanishes while the efficiency approaches the Carnot limit for $\Delta \to \Delta_{cr}$. All the figures are for Lorentzian bath spectra.



(a) Plot of TUR ratio for heat current J_h (green solid line) and power $\mathcal{P}$ (blue dashed-dotted line) with Δ. Inset: Plot of the difference between the TUR ratio for the power $\mathcal{P}$ and the heat current J_h, as a function of Δ.



(b) Plot of $\eta_{\rm ref}^{(2)}$ with Δ. Critical modulation frequency $\Delta_{cr} = 10$. The horizontal red dashed line represents $\eta_R^2 = 1$. Inset: Plot of $\eta_{\rm eng}^{(2)}$ with Δ for the same parameter values. The red dashed horizontal line in the represents $\eta_C^2 = 0.25$.

$$\frac{\mathrm{var}(J_h)}{\langle J_h \rangle^2} = \frac{\mathrm{var}(J_c)}{\langle J_c \rangle^2}, \quad \mathcal{D} := \frac{\mathrm{var}(\mathcal{P})}{\langle \mathcal{P} \rangle^2} - \frac{\mathrm{var}(J_h)}{\langle J_h \rangle^2} = \frac{1}{2}\left(\frac{\omega_0^2}{\Delta^2} - 1 \right). \tag{14}$$

As the occupation probabilities follow a Pauli type master equation with time independent coefficients, in the presence of KMS condition we can expect the validity of TUR,

$$\langle \dot{S} \rangle \frac{\mathrm{var}(\mathcal{P})}{\langle \mathcal{P} \rangle^2} \ge \langle \dot{S} \rangle \frac{\mathrm{var}(J_h)}{\langle J_h \rangle^2} = \langle \dot{S} \rangle \frac{\mathrm{var}(J_c)}{\langle J_c \rangle^2} \ge 2. \tag{15}$$

Another quantification of fluctuation

$$\eta_{\rm eng}^{(2)} = \frac{\mathrm{var}(\mathcal{P})}{\mathrm{var}(J_h)}, \text{ for engine}; \quad \eta_{\rm ref}^{(2)} = \frac{\mathrm{var}(J_c)}{\mathrm{var}(\mathcal{P})}, \text{ for refrigerator}. \tag{16}$$

Based on the Onsager's reciprocity relations for autonomous continuous machine, it was recently shown that $\eta^{(2)}$ is bounded from both above as well as below [2], as

$$\langle \eta \rangle_{\rm eng}^2 \le \eta_{\rm eng}^{(2)} \le \eta_C^2, \text{ for engine}; \quad \langle \eta \rangle_{\rm ref}^2 \le \eta_{\rm ref}^{(2)} \le \left(\frac{1 - \eta_C}{\eta_C} \right)^2 = \eta_R^2, \text{ for refrigerator}. \tag{17}$$

Similar bounds were also shown to exist for finite-time driven quantum Otto engines. In contrast, for the refrigerator case violation of the lower bound was pointed out. For our case, in the engine and refrigerator regime, we arrive at the following relations:

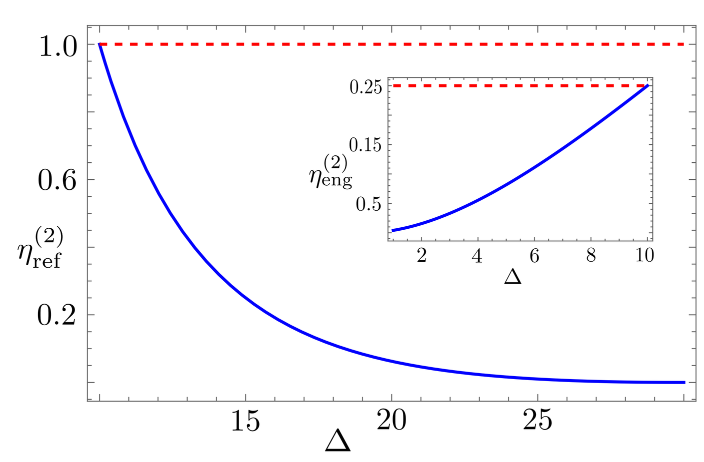
$$\eta_{\rm eng}^{(2)} = \frac{\mathrm{var}(P)}{\mathrm{var}(J_h)} > \frac{\langle P \rangle^2}{\langle J_h \rangle^2} = \langle \eta \rangle_{\rm eng}^2, \quad \eta_{\rm ref}^{(2)} = \frac{\mathrm{var}(J_c)}{\mathrm{var}(P)} < \frac{\langle J_c \rangle^2}{\langle P \rangle^2} = \langle \eta \rangle_{\rm ref}^2 \le \eta_R^2, \tag{18}$$

thus signifying the violation of the lower bound for refrigerator.

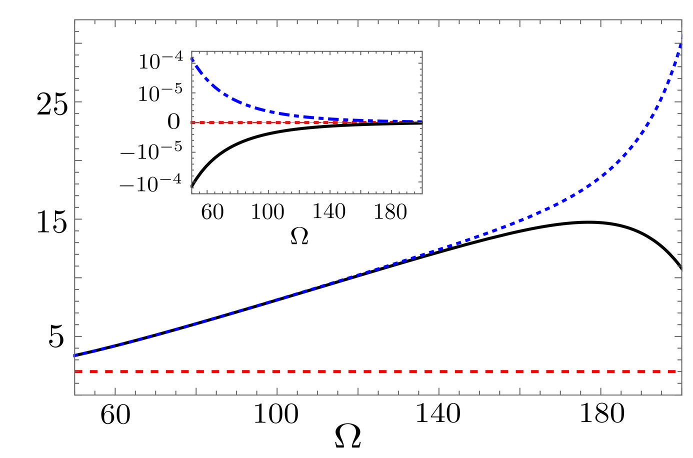
Circular driving

$$H_S(t) = \frac{\omega_0}{2}\sigma_z + g(\sigma^- e^{i\Omega t} + \sigma^+ e^{-i\Omega t}). \tag{19}$$

We use Floquet formalism with full secular approximation to derive the master equation. Again the diagonal and non-diagonal entries of density matrix decouple implying the validity of TUR. The machine is neither a heat engine nor a refrigerator but accelerator.



(c) Plot of $\eta_{\rm ref}^{(2)}$ with Δ for sinusoidal modulation. The horizontal red dashed line represents $\eta_R^2 = 1$. Inset: We plot $\eta_{\rm eng}^{(2)}$ with Δ for the same parameter values. The red dashed horizontal line in the represents $\eta_C^2 = 0.25$.



(d) Plot of TUR ratio for hot (black solid) and cold (blue dotted line) bath currents as a function of driving frequency Ω, for circular modulation.

Conclusion

- We considered a minimal model of periodically modulated continuous quantum thermal machine and analyzed the validity of TUR.
- We studied some other quantification of fluctuation to assess the quality of the machine and proved the violation of its lower bound for refrigerator regime.
- In case of circular modulation also TUR remains valid in presence of full secular approximation.
- It is an future goal to see the validity of TUR with partial secular approximation.

References

[1] D. Gelbwaser-Klimovsky, R. Alicki, and G. Kurizki, *Phys. Rev. E* **87**, 012140 (2013).

[2] S. Saryal, M. Gerry, I. Khait, D. Segal, and B. K. Agarwalla, *Phys. Rev. Lett.* **127**, 190603 (2021).